

# Noninterference for Free

## (Technical Appendix)

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# 1 Source Language: DCC

This section describes the terminating portion of DCC [1], with slightly altered semantics as described by Tse and Zdancewic [3].

## 1.1 Syntax and Dynamic Semantics

DCC is a call-by-name, simply-typed  $\lambda$ -calculus, extended with a lattice of monads which allow dependencies to be expressed in a program. Except for the monadic operations, the language is completely standard. Figure 1 defines the syntax and dynamic semantics for DCC.

<i>Types</i>	$s ::= 1 \mid s_1 \times s_2 \mid s_1 + s_2 \mid s_1 \rightarrow s_2 \mid T_\ell s$
<i>Values</i>	$v ::= x \mid \langle \rangle \mid \langle e_1, e_2 \rangle \mid \text{inj}_i e \mid \lambda x : s. e \mid \eta_\ell e$
<i>Terms</i>	$e ::= v \mid \text{prj}_i e \mid \text{case } e \text{ of } \text{inj}_1 x_1. e_1 \parallel \text{inj}_2 x_2. e_2 \mid \text{bind } x = e_1 \text{ in } e_2 \mid e_1 e_2$
<i>Eval. Contexts</i>	$E ::= [\cdot]s \mid \text{case } E \text{ of } \text{inj}_1 x_1. e_1 \parallel \text{inj}_2 x_2. e_2 \mid \text{bind } x = E \text{ in } e \mid E e$
<i>Lattice</i>	$\mathcal{L} ::= (\mathcal{L}_\ell, \mathcal{L}_\sqsubseteq)$
<i>Lattice Labels</i>	$\mathcal{L}_\ell ::= \cdot \mid \mathcal{L}_\ell, \ell$
<i>Lattice Ordering</i>	$\mathcal{L}_\sqsubseteq ::= \cdot \mid \mathcal{L}_\sqsubseteq, \ell_1 \sqsubseteq \ell_2$

$$\boxed{e \mapsto e'}$$

$$\begin{aligned}
 \text{prj}_i \langle e_1, e_2 \rangle &\mapsto e_i \\
 \text{case } (\text{inj}_i e) \text{ of } \text{inj}_1 x_1. e_1 \parallel \text{inj}_2 x_2. e_2 &\mapsto e_i[e/x_i] \\
 (\lambda x : s. e_1) e_2 &\mapsto e_1[e_2/x] \\
 \text{bind } x = \eta_\ell e_1 \text{ in } e_2 &\mapsto e_2[e_1/x] \\
 \frac{e \mapsto e'}{E[e] \mapsto E[e']} &
 \end{aligned}$$

Figure 1: DCC: Syntax and Dynamic Semantics

## 1.2 Static Semantics

$\ell \sqsubseteq \ell'$   $\ell$  is at or below  $\ell'$  in the lattice  $\mathcal{L}$

$\ell \preceq s$  (type  $s$  is protected at  $\ell$ )

$$\begin{array}{c}
 \text{(P-Unit)} \frac{}{\ell \preceq 1} \qquad \text{(P-Pair)} \frac{\ell \preceq s_1 \quad \ell \preceq s_2}{\ell \preceq s_1 \times s_2} \qquad \text{(P-Fun)} \frac{\ell \preceq s_2}{\ell \preceq s_1 \rightarrow s_2} \\
 \text{(P-Label1)} \frac{\ell \not\sqsubseteq \ell' \quad \ell \preceq s}{\ell \preceq T_{\ell'} s} \qquad \text{(P-Label2)} \frac{\ell \sqsubseteq \ell'}{\ell \preceq T_{\ell'} s}
 \end{array}$$

Figure 2: DCC: Protection Semantics

*Term Environment*  $\Gamma ::= \cdot \mid \Gamma, x : s$

$\Gamma \vdash e : s$

$$\begin{array}{c}
 \text{(DT-Unit)} \frac{}{\Gamma \vdash \langle \rangle : 1} \qquad \text{(DT-Pair)} \frac{\Gamma \vdash e_1 : s_1 \quad \Gamma \vdash e_2 : s_2}{\Gamma \vdash \langle e_1, e_2 \rangle : s_1 \times s_2} \qquad \text{(DT-Prj)} \frac{\Gamma \vdash e : s_1 \times s_2}{\Gamma \vdash \text{prj}_i e : s_i} \\
 \text{(DT-Sum)} \frac{\Gamma \vdash e : s_i}{\Gamma \vdash \text{inj}_i e : s_1 + s_2} \qquad \text{(DT-Case)} \frac{\Gamma \vdash e : s_1 + s_2 \quad \Gamma, x_i : s_i \vdash e_i : s}{\Gamma \vdash \text{case } e \text{ of } \text{inj}_1 x_1. e_1 \parallel \text{inj}_2 x_2. e_2 : s} \\
 \text{(DT-Var)} \frac{(x : s) \in \Gamma}{\Gamma \vdash x : s} \qquad \text{(DT-Fun)} \frac{\Gamma, x : s_1 \vdash e : s_2}{\Gamma \vdash \lambda x : s_1. e : s_1 \rightarrow s_2} \qquad \text{(DT-App)} \frac{\Gamma \vdash e_1 : s_1 \rightarrow s_2 \quad \Gamma \vdash e_2 : s_1}{\Gamma \vdash e_1 e_2 : s_2} \\
 \text{(DT-Prot)} \frac{\Gamma \vdash e : s}{\Gamma \vdash \eta_{\ell} e : T_{\ell} s} \qquad \text{(DT-Bind)} \frac{\Gamma \vdash e_1 : T_{\ell} s_1 \quad \Gamma, x : s_1 \vdash e_2 : s_2 \quad \ell \preceq s_2}{\Gamma \vdash \text{bind } x = e_1 \text{ in } e_2 : s_2}
 \end{array}$$

Figure 3: DCC: Typing Rules

### 1.3 DCC Logical Relation

Figure 4 presents the logical relation for DCC. We define a relation,  $\mathcal{V}[\mathbf{s}]_\zeta$ , which relates closed values at type  $\mathbf{s}$ , a relation  $\mathcal{E}[\mathbf{s}]_\zeta$ , which relates closed expressions, and finally extend it to open terms (written  $\Gamma \vdash e \approx_\zeta e' : \mathbf{s}$ ). Note that each relation is parameterized by an observer  $\zeta$ .

$$\begin{aligned}
Atom[\mathbf{s}] &= \{ (e_1, e_2) \mid \vdash e_1 : \mathbf{s} \wedge \vdash e_2 : \mathbf{s} \} \\
Atom^{\text{val}}[\mathbf{s}] &= \{ (v_1, v_2) \mid \vdash v_1 : \mathbf{s} \wedge \vdash v_2 : \mathbf{s} \} \\
\mathcal{V}[\mathbf{1}]_\zeta &= \{ (\langle \rangle, \langle \rangle) \in Atom^{\text{val}}[\mathbf{1}] \} \\
\mathcal{V}[\mathbf{s} \times \mathbf{s}']_\zeta &= \{ (\langle e_1, e_1' \rangle, \langle e_2, e_2' \rangle) \in Atom^{\text{val}}[\mathbf{s} \times \mathbf{s}'] \mid \\
&\quad (e_1, e_2) \in \mathcal{E}[\mathbf{s}]_\zeta \wedge (e_1', e_2') \in \mathcal{E}[\mathbf{s}']_\zeta \} \\
\mathcal{V}[\mathbf{s} + \mathbf{s}']_\zeta &= \{ (\text{inj}_1 e_1, \text{inj}_1 e_2) \in Atom^{\text{val}}[\mathbf{s} + \mathbf{s}'] \mid (e_1, e_2) \in \mathcal{E}[\mathbf{s}]_\zeta \} \\
&\quad \cup \{ (\text{inj}_2 e_1, \text{inj}_2 e_2) \in Atom^{\text{val}}[\mathbf{s} + \mathbf{s}'] \mid (e_1, e_2) \in \mathcal{E}[\mathbf{s}']_\zeta \} \\
\mathcal{V}[\mathbf{s}' \rightarrow \mathbf{s}]_\zeta &= \{ (\lambda x : \mathbf{s}'. e_1, \lambda x : \mathbf{s}'. e_2) \in Atom^{\text{val}}[\mathbf{s}' \rightarrow \mathbf{s}] \mid \\
&\quad \forall (e_1', e_2') \in \mathcal{E}[\mathbf{s}']_\zeta. (e_1[e_1'/x], e_2[e_2'/x]) \in \mathcal{E}[\mathbf{s}]_\zeta \} \\
\mathcal{V}[\mathbf{T}_\ell \mathbf{s}]_\zeta &= \{ (\eta_\ell e_1, \eta_\ell e_2) \in Atom^{\text{val}}[\mathbf{T}_\ell \mathbf{s}] \mid \ell \sqsubseteq \zeta \implies (e_1, e_2) \in \mathcal{E}[\mathbf{s}]_\zeta \} \\
\mathcal{E}[\mathbf{s}]_\zeta &= \{ (e_1, e_2) \in Atom[\mathbf{s}] \mid \exists v_1, v_2. e_1 \mapsto^* v_1 \wedge e_2 \mapsto^* v_2 \wedge (v_1, v_2) \in \mathcal{V}[\mathbf{s}]_\zeta \} \\
\mathcal{G}[\cdot]_\zeta &= (\emptyset, \emptyset) \\
\mathcal{G}[\Gamma, x : \mathbf{s}]_\zeta &= \{ (\gamma_1[x \mapsto e_1], \gamma_2[x \mapsto e_2]) \mid (\gamma_1, \gamma_2) \in \mathcal{G}[\Gamma]_\zeta \wedge ((e_1, e_2) \in \mathcal{E}[\mathbf{s}]_\zeta) \} \\
\Gamma \vdash e_1 \approx_\zeta e_2 : \mathbf{s} &\stackrel{\text{def}}{=} \Gamma \vdash e_1 : \mathbf{s} \wedge \Gamma \vdash e_2 : \mathbf{s} \wedge \\
&\quad \forall (\gamma_1, \gamma_2) \in \mathcal{G}[\Gamma]_\zeta. (\gamma_1(e_1), \gamma_2(e_2)) \in \mathcal{E}[\mathbf{s}]_\zeta
\end{aligned}$$

Figure 4: DCC: Logical Relation

## 2 Target Language: $F_\omega$

In this section we present the target language  $F_\omega$ , the higher-order polymorphic lambda calculus with unit, pairs, and sums. The language is completely standard (see Pierce [2], Chapter 30).

### 2.1 Syntax and Dynamic Semantics

<i>Kinds</i>	$\kappa ::= * \mid \kappa \rightarrow \kappa$
<i>Types</i>	$t ::= 1 \mid t_1 \times t_2 \mid t_1 + t_2 \mid t_1 \rightarrow t_2 \mid \alpha \mid \forall \alpha :: \kappa. t \mid \lambda \alpha :: \kappa. t \mid t_1 t_2$
<i>Values</i>	$u ::= x \mid \langle \rangle \mid \langle m_1, m_2 \rangle \mid \text{inj}_i m \mid \lambda x :: t. m \mid \Lambda \alpha :: \kappa. m$
<i>Terms</i>	$m ::= u \mid \text{prj}_i m \mid \text{case } m \text{ of } \text{inj}_1 x_1. m_1 \parallel \text{inj}_2 x_2. m_2 \mid m_1 m_2 \mid m [t]$
<i>Eval. Contexts</i>	$E ::= [\cdot]_T \mid \text{prj}_i E \mid \text{case } E \text{ of } \text{inj}_1 x_1. m_1 \parallel \text{inj}_2 x_2. m_2 \mid E m \mid E [t]$

  

$m \mapsto m'$

$$\begin{aligned}
 \text{prj}_i \langle m_1, m_2 \rangle &\mapsto m_i \\
 \text{case } (\text{inj}_i m) \text{ of } \text{inj}_1 y_1. m_1 \parallel \text{inj}_2 y_2. m_2 &\mapsto m_i[m/y_i] \\
 (\lambda x :: t. m_1) m_2 &\mapsto m_1[m_2/x] \\
 (\Lambda \alpha :: \kappa. m) [t] &\mapsto m[t/\alpha] \\
 \frac{m \mapsto m'}{E[m] \mapsto E[m']} &
 \end{aligned}$$

Figure 5:  $F_\omega$ : Syntax and Dynamic Semantics

## 2.2 Static Semantics

$$\boxed{\Delta \vdash t :: \kappa}$$

$$\begin{array}{c}
\text{(FK-Var)} \frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa} \quad \text{(FK-Unit)} \frac{}{\Delta \vdash \mathbf{1} :: *} \quad \text{(FK-Pair)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \times t_2 :: *} \\
\text{(FK-Sum)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *} \quad \text{(FK-Arrow)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \rightarrow t_2 :: *} \quad \text{(FK-Abs)} \frac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha :: \kappa. t :: *} \\
\text{(FK-Fun)} \frac{\Delta, \alpha :: \kappa_1 \vdash t :: \kappa_2}{\Delta \vdash \lambda \alpha :: \kappa. t :: \kappa_1 \rightarrow \kappa_2} \quad \text{(FK-App)} \frac{\Delta \vdash t_1 :: \kappa_1 \rightarrow \kappa_2 \quad \Delta \vdash t_2 :: \kappa_1}{\Delta \vdash t_1 t_2 :: \kappa_2}
\end{array}$$

$$\boxed{\Delta \vdash \Gamma}$$

$$\begin{array}{c}
\text{(FWF-Emp)} \frac{}{\Delta \vdash \cdot} \quad \text{(FWF-Var)} \frac{\Delta \vdash \Gamma \quad \Delta \vdash t :: *}{\Delta \vdash \Gamma, x : t}
\end{array}$$

Figure 6:  $F_\omega$ : Kinding

$$\boxed{t \equiv t'}$$

$$\begin{array}{c}
\text{(FE-Refl)} \frac{}{t \equiv t} \quad \text{(FE-Sym)} \frac{t_1 \equiv t_2}{t_2 \equiv t_1} \quad \text{(FE-Trans)} \frac{t_1 \equiv t_2 \quad t_2 \equiv t_3}{t_1 \equiv t_3} \\
\text{(FE-Pair)} \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 \times t_2 \equiv t'_1 \times t'_2} \quad \text{(FE-Sum)} \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 + t_2 \equiv t'_1 + t'_2} \\
\text{(FE-Arrow)} \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 \rightarrow t_2 \equiv t'_1 \rightarrow t'_2} \quad \text{(FE-All)} \frac{t_1 \equiv t_2}{\forall \alpha :: \kappa. t_1 \equiv \forall \alpha :: \kappa. t_2} \\
\text{(FE-Abs)} \frac{t_1 \equiv t_2}{\lambda \alpha :: \kappa. t_1 \equiv \lambda \alpha :: \kappa. t_2} \quad \text{(FE-App)} \frac{t_1 \equiv t'_1 \quad t_2 \equiv t'_2}{t_1 t_2 \equiv t'_1 t'_2} \quad \text{(FE-Beta)} \frac{}{(\lambda \alpha :: \kappa. t_1) t_2 \equiv t_1[t_2/\alpha]}
\end{array}$$

Figure 7:  $F_\omega$ : Type Equivalence

*Type Environment*  $\Delta ::= \cdot \mid \Delta, \alpha :: \kappa$   
*Term Environment*  $\Gamma ::= \cdot \mid \Gamma, x : t$

$\Delta; \Gamma \vdash m : t$

$$\begin{array}{c}
\text{(FT-Unit)} \frac{\Delta \vdash \Gamma}{\Delta; \Gamma \vdash \langle \rangle : \mathbf{1}} \quad
\text{(FT-Var)} \frac{x : t \in \Gamma \quad \Delta \vdash \Gamma}{\Delta; \Gamma \vdash x : t} \quad
\text{(FT-Pair)} \frac{\Delta; \Gamma \vdash m_1 : t_1 \quad \Delta; \Gamma \vdash m_2 : t_2}{\Delta; \Gamma \vdash \langle m_1, m_2 \rangle : t_1 \times t_2} \\
\\
\text{(FT-Prj)} \frac{\Delta; \Gamma \vdash m : t_1 \times t_2}{\Delta; \Gamma \vdash \text{prj}_i m : t_i} \\
\\
\text{(FT-Sum)} \frac{\Delta; \Gamma \vdash m : t_i}{\Delta; \Gamma \vdash \text{inj}_i m : t_1 + t_2} \\
\\
\text{(FT-Case)} \frac{\Delta; \Gamma \vdash m : t_1 + t_2 \quad \Delta; \Gamma, x : t_1 \vdash m_1 : t \quad \Delta; \Gamma, x : t_2 \vdash m_2 : t}{\Delta; \Gamma \vdash \text{case } m \text{ of } \text{inj}_1 x. m_1 \parallel \text{inj}_2 x. m_2 : t} \\
\\
\text{(FT-Fun)} \frac{\Delta; \Gamma, x : t_1 \vdash m : t_2 \quad \Delta \vdash t_1 :: *}{\Delta; \Gamma \vdash \lambda x : t_1. m : t_1 \rightarrow t_2} \quad
\text{(FT-App)} \frac{\Delta; \Gamma \vdash m_1 : t_1 \rightarrow t_2 \quad \Delta; \Gamma \vdash m_2 : t_1}{\Delta; \Gamma \vdash m_1 m_2 : t_2} \\
\\
\text{(FT-Abs)} \frac{\Delta, \alpha :: \kappa; \Gamma \vdash m : t}{\Delta; \Gamma \vdash \Lambda \alpha :: \kappa. m : \forall \alpha :: \kappa. t} \quad
\text{(FT-Inst)} \frac{\Delta; \Gamma \vdash m : \forall \alpha :: \kappa. t_1 \quad \Delta \vdash t_2 :: \kappa}{\Delta; \Gamma \vdash m [t_2] : t_1[t_2/\alpha]} \\
\\
\text{(FT-Eqv)} \frac{\Delta; \Gamma \vdash m : t_1 \quad t_1 \equiv t_2 \quad \Delta \vdash t_2 :: *}{\Delta; \Gamma \vdash m : t_2}
\end{array}$$

Figure 8:  $F_\omega$ : Typing

## 2.3 $F_\omega$ Logical Relation

In this subsection we present an open logical relation for  $F_\omega$ .

$$\begin{aligned}
Atom [t_1, t_2]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\mathbf{m}_1, \mathbf{m}_2) \mid \mathbf{D} \vdash t_1 \wedge \mathbf{D} \vdash t_2 \wedge \mathbf{D}; \mathbf{G} \vdash \mathbf{m}_1 : t_1 \wedge \mathbf{D}; \mathbf{G} \vdash \mathbf{m}_2 : t_2 \} \\
Atom [t]_{\rho}^{\mathbf{D};\mathbf{G}} &= Atom [\rho_1(t), \rho_2(t)]_{\rho}^{\mathbf{D};\mathbf{G}} \\
Rel_*^{\mathbf{D};\mathbf{G}} &= \{ (t_1, t_2, \mathbf{R}) \mid \mathbf{R} \subseteq Atom [t_1, t_2]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
Rel_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} &= \{ (t_1, t_2, \mathbf{R}) \mid \\
&\quad (\forall \pi \in Rel_{\kappa_1}^{\mathbf{D};\mathbf{G}}. (t_1 \pi_1, t_2 \pi_2, (\mathbf{R} \pi)) \in Rel_{\kappa_2}^{\mathbf{D};\mathbf{G}} \wedge \\
&\quad (\forall \pi' \in Rel_{\kappa_1}^{\mathbf{D};\mathbf{G}}. \pi \equiv_{\kappa_1}^{\mathbf{D};\mathbf{G}} \pi' \implies \\
&\quad \quad \mathbf{R} \pi \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathbf{R} \pi') \} \\
\rho \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \rho' &\stackrel{\text{def}}{=} \forall \alpha :: \kappa \in \text{dom}(\rho). \rho(\alpha) \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \rho'(\alpha) \\
\pi \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \pi &\stackrel{\text{def}}{=} \pi_1 \equiv \pi'_1 \wedge \pi_2 \equiv \pi'_2 \wedge \pi_{\mathbf{R}} \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \pi'_{\mathbf{R}} \\
\mathbf{R} \equiv_*^{\mathbf{D};\mathbf{G}} \mathbf{R}' &\stackrel{\text{def}}{=} \forall \mathbf{m}_1, \mathbf{m}_2. (\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R} \iff (\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R}' \\
\mathbf{R} \equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} \mathbf{R}' &\stackrel{\text{def}}{=} \forall \pi \in Rel_{\kappa_1}^{\mathbf{D};\mathbf{G}}. \mathbf{R} \pi \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathbf{R}' \pi \\
\mathcal{T} [t :: *]_{\rho}^{\mathbf{D};\mathbf{G}} &= \mathcal{V} [t]_{\rho}^{\mathbf{D};\mathbf{G}} \\
\mathcal{T} [\alpha :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} &= \rho_{\mathbf{R}}(\alpha) \\
\mathcal{T} [\lambda \alpha :: \kappa_1. t :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} &= \lambda_{\mathbf{R}} \pi. \{ \mathcal{T} [t :: \kappa_2]_{\rho[\alpha :: \kappa_1 \mapsto \pi]}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{T} [t_1 t_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} &= (\mathcal{T} [t_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \\
&\quad (\rho_1 t_2, \rho_2 t_2, \mathcal{T} [t_2 :: \kappa_1]_{\rho}^{\mathbf{D};\mathbf{G}}))
\end{aligned}$$

Figure 9:  $F_\omega$ : Logical Relations (Higher-Order)



$$\begin{aligned}
Atom^{\text{val}} [t_1, t_2]^{\mathbf{D};\mathbf{G}} &= \{ (u_1, u_2) \mid \mathbf{D} \vdash t_1 \wedge \mathbf{D} \vdash t_2 \wedge \mathbf{D}; \mathbf{G} \vdash u_1 : t_1 \wedge \mathbf{D}; \mathbf{G} \vdash u_2 : t_2 \} \\
Atom^{\text{val}} [t]_{\rho}^{\mathbf{D};\mathbf{G}} &= Atom^{\text{val}} [\rho_1(t), \rho_2(t)]^{\mathbf{D};\mathbf{G}} \\
\mathcal{V}[\alpha]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (m_1, m_2) \in \rho_R(\alpha) \} \\
\mathcal{V}[1]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\langle \rangle, \langle \rangle) \in Atom^{\text{val}} [1]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{V}[t \times t']_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\langle m_1, m'_1 \rangle, \langle m_2, m'_2 \rangle) \in Atom^{\text{val}} [t \times t']_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
&\quad (m_1, m_2) \in \mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}} \wedge (m'_1, m'_2) \in \mathcal{E}[t']_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{V}[t + t']_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\text{inj}_1 m_1, \text{inj}_1 m_2) \in Atom^{\text{val}} [t + t']_{\rho}^{\mathbf{D};\mathbf{G}} \mid (m_1, m_2) \in \mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
&\cup \{ (\text{inj}_2 m_1, \text{inj}_2 m_2) \in Atom^{\text{val}} [t + t']_{\rho}^{\mathbf{D};\mathbf{G}} \mid (m_1, m_2) \in \mathcal{E}[t']_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{V}[t' \rightarrow t]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\lambda x : \rho_1(t').m_1, \lambda x : \rho_2(t').m_2) \in Atom^{\text{val}} [t' \rightarrow t]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
&\quad \forall (m'_1, m'_2) \in \mathcal{E}[t']_{\rho}^{\mathbf{D};\mathbf{G}}. (m_1[m'_1/x], m_2[m'_2/x]) \in \mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{V}[\forall \alpha :: \kappa. t]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\Lambda \alpha :: \kappa. m_1, \Lambda \alpha :: \kappa. m_2) \in Atom^{\text{val}} [\forall \alpha :: \kappa. t]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
&\quad \forall \pi \in Rel_{\kappa}^{\mathbf{D};\mathbf{G}}. \\
&\quad (m_1[\pi_1/\alpha], m_2[\pi_2/\alpha]) \in \mathcal{E}[t]_{\rho[\alpha :: \kappa \mapsto \pi]}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{V}[t_1 t_2]_{\rho}^{\mathbf{D};\mathbf{G}} &= \mathcal{T}[t_1 t_2 :: *]_{\rho}^{\mathbf{D};\mathbf{G}} \\
\mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (m_1, m_2) \in Atom[t]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
&\quad \exists m'_1, m'_2. m_1 \mapsto^* m'_1 \wedge m_2 \mapsto^* m'_2 \wedge \\
&\quad \text{irred}(m'_1) \wedge \text{irred}(m'_2) \wedge \\
&\quad (m'_1, m'_2) \in \mathcal{V}[t]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{D}[\cdot]_{\rho}^{\mathbf{D};\mathbf{G}} &= \emptyset \\
\mathcal{D}[\Delta, \alpha :: \kappa]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ \rho[\alpha :: \kappa \mapsto \pi] \mid \rho \in \mathcal{D}[\Delta]_{\rho}^{\mathbf{D};\mathbf{G}} \wedge \pi \in Rel_{\kappa}^{\mathbf{D};\mathbf{G}} \} \\
\mathcal{G}[\cdot]_{\rho}^{\mathbf{D};\mathbf{G}} &= (\emptyset, \emptyset) \\
\mathcal{G}[\Gamma, x : t]_{\rho}^{\mathbf{D};\mathbf{G}} &= \{ (\gamma_1[x \mapsto m_1], \gamma_2[x \mapsto m_2]) \mid \\
&\quad (\gamma_1, \gamma_2) \in \mathcal{G}[\Gamma]_{\rho}^{\mathbf{D};\mathbf{G}} \wedge (m_1, m_2) \in \mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}} \} \\
\Delta; \Gamma \vdash m_1 \approx m_2 : t &\stackrel{\text{def}}{=} \Delta; \Gamma \vdash m_1 : t \wedge \Delta; \Gamma \vdash m_2 : t \wedge \\
&\quad \forall \mathbf{D}, \mathbf{G}, \rho, \gamma_1, \gamma_2. \text{dom}(\mathbf{D}) \# \text{dom}(\Delta) \wedge \text{dom}(\mathbf{G}) \# \text{dom}(\Gamma) \wedge \\
&\quad \rho \in \mathcal{D}[\Delta]_{\rho}^{\mathbf{D};\mathbf{G}} \wedge (\gamma_1, \gamma_2) \in \mathcal{G}[\Gamma]_{\rho}^{\mathbf{D};\mathbf{G}} \implies \\
&\quad (\rho_1 \gamma_1(m_1), \rho_2 \gamma_2(m_2)) \in \mathcal{E}[t]_{\rho}^{\mathbf{D};\mathbf{G}}
\end{aligned}$$

Figure 10:  $F_{\omega}$ : Logical Relation

### 3 Translation: DCC to $F_\omega$

In this section, we present the type-directed translation from DCC to  $F_\omega$ .

$$\begin{aligned}\mathcal{L}_\ell^+ &= \{\alpha_\ell :: * \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_\preceq :: * \rightarrow * \rightarrow *\} \\ \mathcal{L}_\sqsubseteq^+ &= \{\mathbf{c}_{\ell\ell'} : \alpha_{\ell'} \rightarrow \alpha_\ell \mid \ell \sqsubseteq \ell' \in \mathcal{L}_\sqsubseteq\}\end{aligned}$$

$\boxed{s^+}$  where  $\mathcal{L}_\ell^+ \vdash s^+ :: *$

$$\begin{aligned}1^+ &= \mathbf{1} \\ (s_1 \times s_2)^+ &= s_1^+ \times s_2^+ \\ (s_1 + s_2)^+ &= s_1^+ + s_2^+ \\ (s_1 \rightarrow s_2)^+ &= s_1^+ \rightarrow s_2^+ \\ (\mathbb{T}_\ell s)^+ &= \forall \beta :: *. ((\alpha_\preceq \alpha_\ell \beta) \times (s^+ \rightarrow \beta)) \rightarrow \beta\end{aligned}$$

$\boxed{\Gamma^+}$

$$\begin{aligned}(\cdot)^+ &= \cdot \\ (\Gamma, \mathbf{x} : s)^+ &= \Gamma^+, \mathbf{x} : s^+\end{aligned}$$

Figure 11: DCC to  $F_\omega$ : Lattice, Type, and Context Translation

We will often simply write  $(\mathbb{T}_\ell t)^+$  to mean  $\forall \beta :: *. ((\alpha_\preceq \alpha_\ell \beta) \times (t \rightarrow \beta)) \rightarrow \beta$ . Of course,  $(\mathbb{T}_\ell t)^+$  doesn't make sense since  $t$  is already a target type, so read this as a macro for  $\forall \beta :: *. ((\alpha_\preceq \alpha_\ell \beta) \times (t \rightarrow \beta)) \rightarrow \beta$ .

$\boxed{\lambda^+}$  proof constructors

$$\begin{aligned}
\mathbf{p}_1 & : \forall \beta_\ell :: *. (\alpha_\leq \beta_\ell \mathbf{1}), \\
\mathbf{p}_\times & : \forall \beta_\ell :: *. \forall \alpha_1 :: *. \forall \alpha_2 :: *. \\
& \quad ((\alpha_\leq \beta_\ell \alpha_1) \times (\alpha_\leq \beta_\ell \alpha_2)) \rightarrow (\alpha_\leq \beta_\ell (\alpha_1 \times \alpha_2)), \\
\mathbf{p}_\rightarrow & : \forall \beta_\ell :: *. \forall \alpha_1 :: *. \forall \alpha_2 :: *. \\
& \quad (\alpha_\leq \beta_\ell \alpha_2) \rightarrow (\alpha_\leq \beta_\ell (\alpha_1 \rightarrow \alpha_2)), \\
\mathbf{p}_{T_1} & : \forall \beta_\ell :: *. \forall \beta_{\ell'} :: *. \forall \alpha :: *. \\
& \quad (\alpha_\leq \beta_\ell \alpha) \rightarrow \\
& \quad (\alpha_\leq \beta_\ell (\forall \beta :: *. ((\alpha_\leq \beta_{\ell'} \beta) \times (\alpha \rightarrow \beta)) \rightarrow \beta)) \\
\mathbf{p}_{T_2} & : \forall \beta_\ell :: *. \forall \beta_{\ell'} :: *. \forall \alpha :: *. \\
& \quad (\beta_{\ell'} \rightarrow \beta_\ell) \rightarrow \\
& \quad (\alpha_\leq \beta_\ell (\forall \beta :: *. ((\alpha_\leq \beta_{\ell'} \beta) \times (\alpha \rightarrow \beta)) \rightarrow \beta)),
\end{aligned}$$

$\boxed{\mathbf{pf}[\ell \preceq s] : (\alpha_\preceq \alpha_\ell s^+)}$  proof-term construction

$$\begin{aligned}
\mathbf{pf}[\ell \preceq \mathbf{1}] & \stackrel{\text{def}}{=} \mathbf{p}_1[\alpha_\ell] \\
\mathbf{pf}[\ell \preceq s_1 \times s_2] & \stackrel{\text{def}}{=} \mathbf{p}_\times[\alpha_\ell][s_1^+][s_2^+] \langle \mathbf{pf}[\ell \preceq s_1], \mathbf{pf}[\ell \preceq s_2] \rangle \\
\mathbf{pf}[\ell \preceq s_1 \rightarrow s_2] & \stackrel{\text{def}}{=} \mathbf{p}_\rightarrow[\alpha_\ell][s_1^+][s_2^+] \mathbf{pf}[\ell \preceq s_2] \\
\mathbf{pf}[\ell \preceq T_{\ell'} s] & \stackrel{\text{def}}{=} \begin{cases} \mathbf{p}_{T_1}[\alpha_\ell][\alpha_{\ell'}][s^+] \mathbf{pf}[\ell \preceq s] & \text{if } \ell \preceq s \text{ and } \ell \not\sqsubseteq \ell' \\ \mathbf{p}_{T_2}[\alpha_\ell][\alpha_{\ell'}][s^+] c_{\ell'\ell} & \text{if } \ell \sqsubseteq \ell' \end{cases}
\end{aligned}$$

Figure 12:  $F_\omega$ : Protection Proofs

We use the name  $\beta_\ell$  to allude to the globally quantified  $\alpha_\ell \mid \ell \in \mathcal{L}_\ell$ , as this type should always be instantiated with some  $\alpha_\ell$ . We will enforce this property via parametricity later.

$\boxed{\Gamma \vdash e : s \rightsquigarrow m}$  where  $\mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+, \Gamma^+ \vdash m : s^+$

$$\begin{array}{c}
\text{(DF-Unit)} \frac{}{\Gamma \vdash \langle \rangle : \mathbf{1} \rightsquigarrow \langle \rangle} \quad \text{(DF-Pair)} \frac{\Gamma \vdash e_1 : s_1 \rightsquigarrow \mathbf{m}_1 \quad \Gamma \vdash e_2 : s_2 \rightsquigarrow \mathbf{m}_2}{\Gamma \vdash \langle e_1, e_2 \rangle : s_1 \times s_2 \rightsquigarrow \langle \mathbf{m}_1, \mathbf{m}_2 \rangle} \quad \text{(DF-Prj)} \frac{\Gamma \vdash e : s_1 \times s_2 \rightsquigarrow \mathbf{m}}{\Gamma \vdash \text{prj}_i e : s_i \rightsquigarrow \text{prj}_i \mathbf{m}} \\
\\
\text{(DF-Sum)} \frac{\Gamma \vdash e : s_i \rightsquigarrow \mathbf{m}}{\Gamma \vdash \text{inj}_i e : s_1 + s_2 \rightsquigarrow \text{inj}_i \mathbf{m}} \\
\\
\text{(DF-Case)} \frac{\Gamma \vdash e : s_1 + s_2 \rightsquigarrow \mathbf{m} \quad \Gamma, x : s_1 \vdash e_1 : s \rightsquigarrow \mathbf{m}_1 \quad \Gamma, x : s_2 \vdash e_2 : s \rightsquigarrow \mathbf{m}_2}{\Gamma \vdash \text{case } e \text{ of } \text{inj}_1 x. e_1 \parallel \text{inj}_2 x. e_2 : s \rightsquigarrow \text{case } \mathbf{m} \text{ of } \text{inj}_1 x. \mathbf{m}_1 \parallel \text{inj}_2 x. \mathbf{m}_2} \\
\\
\text{(DF-Var)} \frac{(x : s) \in \Gamma}{\Gamma \vdash x : s \rightsquigarrow \mathbf{x}} \quad \text{(DF-Fun)} \frac{\Gamma, x : s \vdash e : s_2 \rightsquigarrow \mathbf{m}}{\Gamma \vdash \lambda x : s_1. e : s_1 \rightarrow s_2 \rightsquigarrow \lambda x : s_1^+. \mathbf{m}} \\
\\
\text{(DF-App)} \frac{\Gamma \vdash e_1 : s_1 \rightarrow s_2 \rightsquigarrow \mathbf{m}_1 \quad \Gamma \vdash e_2 : s_1 \rightsquigarrow \mathbf{m}_2}{\Gamma \vdash e_1 e_2 : s_2 \rightsquigarrow \mathbf{m}_1 \mathbf{m}_2} \\
\\
\text{(DF-Prot)} \frac{\Gamma \vdash e : s \rightsquigarrow \mathbf{m}}{\Gamma \vdash \eta_\ell e : \mathbb{T}_\ell s \rightsquigarrow \Lambda \beta :: * \lambda x : ((\alpha_\preceq \alpha_\ell \beta) \times (s^+ \rightarrow \beta)). ((\text{prj}_2 x) \mathbf{m})} \\
\\
\text{(DF-Bind)} \frac{\Gamma \vdash e_1 : \mathbb{T}_\ell s_1 \rightsquigarrow \mathbf{m}_1 \quad \Gamma, x : s_1 \vdash e_2 : s_2 \rightsquigarrow \mathbf{m}_2 \quad \ell \preceq s_2}{\Gamma \vdash \text{bind } x = e_1 \text{ in } e_2 : s_2 \rightsquigarrow \mathbf{m}_1 [(s_2)^+] \langle \text{pf}[\ell \preceq s_2], (\lambda x : s_1^+. \mathbf{m}_2) \rangle}
\end{array}$$

Figure 13: DCC to  $F_\omega$ : Term Translation

### 3.1 Definition of open contexts

$$\begin{aligned}
\mathbf{D}_\ell &= \{\hat{\alpha}_\ell :: * \mid \ell \in \mathcal{L}_\ell\} \cup \{\hat{\alpha}_\leq :: * \rightarrow * \rightarrow *\} \\
\mathbf{G}_\ell &= \{\hat{x}_\ell : \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \\
\mathbf{G}_\leq &= \{\hat{p}_1 : \forall \beta :: *. (\hat{\alpha}_\leq \beta \mathbf{1}), \\
&\quad \hat{p}_\times : \forall \beta_\ell :: *. \forall \alpha_1 :: *. \forall \alpha_2 :: *. ((\hat{\alpha}_\leq \beta_\ell \alpha_1) \times (\hat{\alpha}_\leq \beta_\ell \alpha_2)) \rightarrow (\hat{\alpha}_\leq \beta_\ell (\alpha_1 \times \alpha_2)), \\
&\quad \hat{p}_\rightarrow : \forall \beta_\ell :: *. \forall \alpha_1 :: *. \forall \alpha_2 :: *. (\hat{\alpha}_\leq \beta_\ell \alpha_2) \rightarrow (\hat{\alpha}_\leq \beta_\ell (\alpha_1 \rightarrow \alpha_2)), \\
&\quad \hat{p}_{T_1} : \forall \beta_\ell :: *. \forall \beta_{\ell'} :: *. (\hat{\alpha}_\leq \beta_{\ell'} t) \rightarrow (\hat{\alpha}_\leq \hat{\alpha}_\ell (\forall \beta :: *. ((\alpha_\leq \beta_{\ell'} \beta) \times (t \rightarrow \beta)) \rightarrow \beta)) \\
&\quad \hat{p}_{T_2} : \forall \beta_\ell :: *. \forall \beta_{\ell'} :: *. \forall \alpha :: *. (\beta_{\ell'} \rightarrow \beta_\ell) \rightarrow (\hat{\alpha}_\leq \beta_\ell (\forall \beta :: *. ((\alpha_\leq \beta_{\ell'} \beta) \times (\alpha \rightarrow \beta)) \rightarrow \beta)), \\
\mathbf{\Sigma} &= \mathbf{D}_\ell; \mathbf{G}_\ell, \mathbf{G}_\leq \\
\mathbf{\Sigma}_D &= \mathbf{D}_\ell \\
\mathbf{\Sigma}_G &= \mathbf{G}_\ell, \mathbf{G}_\leq
\end{aligned}$$

Figure 14: Definition of open context

$$\llbracket \mathcal{L}_{\sqsubseteq}^+ \rrbracket = \{ \mathbf{c}_{\ell\ell'} \mapsto \lambda \mathbf{x} : \hat{\alpha}_{\ell} . \hat{\mathbf{x}}_{\ell'} \mid \ell' \sqsubseteq \ell \in \mathcal{L}_{\sqsubseteq} \}$$

Figure 15: Interpretation of Coercions

$$\llbracket \mathcal{L}_{\sqsubseteq}^+ \rrbracket = \left\{ \begin{array}{l} \mathbf{p}_1 \mapsto \Lambda \beta_{\ell} :: * . \hat{\mathbf{p}}_1 [\beta_{\ell}] [1], \\ \mathbf{p}_{\times} \mapsto \Lambda \beta_{\ell} :: * . \Lambda \alpha_1 :: * . \Lambda \alpha_2 :: * . \lambda \mathbf{x} : ((\hat{\alpha}_{\leq} \beta_{\ell} \alpha_1) \times (\hat{\alpha}_{\leq} \beta_{\ell} \alpha_2)) . \hat{\mathbf{p}}_{\times} [\beta_{\ell}] [\alpha_1] [\alpha_2] \mathbf{x}, \\ \mathbf{p}_{\rightarrow} \mapsto \Lambda \beta_{\ell} :: * . \Lambda \alpha_1 :: * . \Lambda \alpha_2 :: * . \lambda \mathbf{x} : (\hat{\alpha}_{\leq} \beta_{\ell} \alpha_2) . \hat{\mathbf{p}}_{\rightarrow} [\beta_{\ell}] [\alpha_1] [\alpha_2] \mathbf{x}, \\ \mathbf{p}_{\mathbf{T}_1} \mapsto \Lambda \beta_{\ell} :: * . \Lambda \beta_{\ell'} :: * . \Lambda \alpha :: * . \lambda \mathbf{x} : (\hat{\alpha}_{\leq} \beta_{\ell} \alpha) . \hat{\mathbf{p}}_{\mathbf{T}_1} [\beta_{\ell}] [\beta_{\ell'}] \mathbf{x} \\ \mathbf{p}_{\mathbf{T}_2} \mapsto \Lambda \beta_{\ell} :: * . \Lambda \beta_{\ell'} :: * . \Lambda \alpha :: * . \lambda \mathbf{x} : (\beta_{\ell'} \rightarrow \beta_{\ell}) . \hat{\mathbf{p}}_{\mathbf{T}_2} [\beta_{\ell}] [\beta_{\ell'}] [\alpha] \mathbf{x}, \end{array} \right.$$

Figure 16: Interpretation of Proof Constructors

### 3.2 Observer-sensitive $F_\omega$ relation

$$\begin{aligned} \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma &= \{ \alpha_\ell :: * \mapsto (\hat{\alpha}_\ell, \hat{\alpha}_\ell, \text{Atom}[\hat{\alpha}_\ell, \hat{\alpha}_\ell]^\Sigma) \} \\ \cup & \left\{ \begin{array}{l} \alpha_\leq :: * \rightarrow * \rightarrow * \mapsto \\ (\lambda\beta_\ell :: *. \lambda\beta :: *. (\hat{\alpha}_\leq \beta_\ell \beta), \lambda\beta_\ell :: *. \lambda\beta :: *. (\hat{\alpha}_\leq \beta_\ell \beta), \\ \lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_\ell). \lambda_R(\mathbf{t}'_1, \mathbf{t}'_2, \mathbf{R}_\beta). \\ \{(\mathbf{m}_1, \mathbf{m}_2) \in \text{Atom}[(\hat{\alpha}_\leq \mathbf{t}_1 \mathbf{t}'_1), (\hat{\alpha}_\leq \mathbf{t}_2 \mathbf{t}'_2)]^\Sigma \mid \mathbf{t}_1 = \hat{\alpha}_\ell \wedge \mathbf{t}_2 = \hat{\alpha}_\ell \wedge \\ \exists s_1^+. \mathbf{t}'_1 = s_1^+ \wedge \ell \preceq s_1 \wedge \\ \exists s_2^+. \mathbf{t}'_2 = s_2^+ \wedge \ell \preceq s_2 \wedge \\ (\ell \not\preceq \zeta \implies \mathbf{R}_\beta = \text{Atom}[\mathbf{t}'_1, \mathbf{t}'_2]^\Sigma)\} \end{array} \right\} \end{aligned}$$

Figure 17:  $F_\omega$ : Relation Interpretation of Lattice and  $\ell \preceq s$

$$\begin{aligned} \mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+, \Gamma^+ \vdash \mathbf{m}_1 \approx_\zeta \mathbf{m}_2 : s^+ &\stackrel{\text{def}}{=} \mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+, \Gamma^+ \vdash \mathbf{m}_1 : s^+ \wedge \mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+, \Gamma^+ \vdash \mathbf{m}_2 : s^+ \wedge \\ &\forall \rho, \gamma_\square, \gamma_\preceq, \gamma_1, \gamma_2. \rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma \wedge \gamma_\square = \llbracket \mathcal{L}_\square^+ \rrbracket \wedge \gamma_\preceq = \llbracket \preceq^+ \rrbracket \wedge \\ &(\gamma_1, \gamma_2) \in \mathcal{G}[\Gamma^+]_\rho^\Sigma \implies \\ &(\rho_1 \gamma_\square \gamma_\preceq \gamma_1(\mathbf{m}_1), \rho_2 \gamma_\square \gamma_\preceq \gamma_2(\mathbf{m}_2)) \in \mathcal{E}[s^+]_\rho^\Sigma \end{aligned}$$

Figure 18:  $F_\omega$ : Observer-sensitive Logical Relation

### 3.3 DCC to $F_\omega$ Logical Relation

In this subsection we define an open, cross-language logical relation, relating DCC to  $F_\omega$ .

$$\begin{aligned}
\eta_k^{\ell,s} &\stackrel{\text{def}}{=} \lambda y:s^+. \Lambda \beta::*. \lambda x:((\alpha_{\leq} \ \alpha_{\ell} \ \beta) \times (s^+ \rightarrow \beta)). ((\text{prj}_2 \ x) \ y) \\
Atom^+[s]_{\delta} &= \{(e, m) \mid \cdot \vdash e : s \wedge \Sigma_D \vdash \delta(s^+) \wedge \Sigma_D; \Sigma_G \vdash m : \delta(s^+)\} \\
Atom^{+val}[s]_{\delta} &= \{(v, u) \mid \cdot \vdash v : s \wedge \Sigma_D \vdash \delta(s^+) \wedge \Sigma_D; \Sigma_G \vdash u : \delta(s^+)\} \\
\mathcal{V}_{\zeta}^+[\mathbf{1}]_{\delta} &= \{(\langle \rangle, \langle \rangle) \in Atom^{+val}[\mathbf{1}]_{\delta}\} \\
\mathcal{V}_{\zeta}^+[s \times s']_{\delta} &= \{(\langle e, e' \rangle, \langle m, m' \rangle) \in Atom^{+val}[s \times s']_{\delta} \mid \\
&\quad (e, m) \in \mathcal{E}_{\zeta}^+[s]_{\delta} \wedge (e', m') \in \mathcal{E}_{\zeta}^+[s']_{\delta}\} \\
\mathcal{V}_{\zeta}^+[s + s']_{\delta} &= \{(\text{inj}_1 e, \text{inj}_1 m) \in Atom^{+val}[s + s']_{\delta} \mid (e, m) \in \mathcal{E}_{\zeta}^+[s]_{\delta}\} \\
&\cup \{(\text{inj}_2 e, \text{inj}_2 m) \in Atom^{+val}[s + s']_{\delta} \mid (e, m) \in \mathcal{E}_{\zeta}^+[s']_{\delta}\} \\
\mathcal{V}_{\zeta}^+[s' \rightarrow s]_{\delta} &= \{(\lambda x:s'. e, \lambda x:\delta((s')^+). m) \in Atom^{+val}[s' \rightarrow s]_{\delta} \mid \\
&\quad \forall e', m'. (e', m') \in \mathcal{E}_{\zeta}^+[s']_{\delta} \implies (e[e'/x], m[m'/x]) \in \mathcal{E}_{\zeta}^+[s]_{\delta}\} \\
\mathcal{V}_{\zeta}^+[\mathbb{T}_{\ell} s]_{\delta} &= \{(\eta_{\ell} e, \Lambda \beta::*. m) \in Atom^{+val}[\mathbb{T}_{\ell} s]_{\delta} \mid \\
&\quad \text{Let } \rho = \llbracket \mathcal{L}_{\ell}^+ \rrbracket_{\zeta}^{\Sigma}. \\
&\quad \exists m'. \Sigma_D; \Sigma_G \vdash m' : \delta(s^+) \wedge \\
&\quad (m[(\mathbb{T}_{\ell} s)^+/\beta] \langle \text{pf}[\llbracket \ell \leq \mathbb{T}_{\ell} s \rrbracket, \eta_k^{\ell,s}], \eta_k^{\ell,s} m'] \in \mathcal{E}[(\mathbb{T}_{\ell} s)^+]_{\rho}^{\Sigma} \wedge \\
&\quad (e, m') \in \mathcal{E}_{\zeta}^+[s^+]_{\delta}\} \\
\mathcal{E}_{\zeta}^+[s]_{\delta} &= \{(e, m) \in Atom^+[s]_{\delta} \mid \\
&\quad \exists v, u. e \mapsto^* v \wedge m \mapsto^* u \wedge (v, u) \in \mathcal{V}_{\zeta}^+[s]_{\delta}\} \\
\mathcal{G}_{\zeta}^+[\cdot]_{\delta} &= \{(\emptyset, \emptyset)\} \\
\mathcal{G}_{\zeta}^+[\Gamma, x : s]_{\delta} &= \{(\gamma[x \mapsto e], \gamma[x \mapsto m]) \mid (\gamma, \gamma) \in \mathcal{G}_{\zeta}^+[\Gamma]_{\delta} \wedge (e, m) \in \mathcal{E}_{\zeta}^+[s]_{\delta}\} \\
\Gamma \mid \Sigma \vdash e \simeq m : s \mid \delta &\stackrel{\text{def}}{=} \Gamma \vdash e : s \wedge \Sigma_D; \Sigma_G, \hat{\Gamma}^+ \vdash m : \delta(s^+) \wedge \\
&\quad \forall \zeta, \gamma, \gamma'. (\gamma, \gamma') \in \mathcal{G}_{\zeta}^+[\Gamma]_{\delta}. \implies \\
&\quad (\gamma(e), \delta(\gamma(m))) \in \mathcal{E}_{\zeta}^+[s]_{\delta} \\
\Gamma \vdash e \simeq m : s &\stackrel{\text{def}}{=} \Gamma \vdash e : s \wedge \mathcal{L}_{\ell}^+; \mathcal{L}_{\zeta}^+, \leq^+, \Gamma^+ \vdash m : s^+ \wedge \\
&\quad (\forall \delta, \gamma_{\zeta}, \gamma_{\leq}. \\
&\quad \delta = \{\alpha_{\ell} \mapsto \hat{\alpha}_{\ell} \mid \ell \in \mathcal{L}_{\ell}\} \cup \{\alpha_{\leq} \mapsto \hat{\alpha}_{\leq}\} \wedge \\
&\quad \gamma_{\zeta} = \llbracket \mathcal{L}_{\zeta}^+ \rrbracket \wedge \gamma_{\leq} = \llbracket \leq^+ \rrbracket \implies \\
&\quad \Gamma \mid \Sigma \vdash e \simeq \delta(\gamma_{\zeta}(\gamma_{\leq}(m))) : s \mid \delta)
\end{aligned}$$

Figure 19: Logical Relation between DCC and  $F_{\omega}$



## 4 Back-translation

In this section, we present a back-translation relation between target terms of translation type and source terms. We need this back-translation when proving that the translation is equivalence preserving.

### 4.1 Context typing

$$\boxed{\Delta; \Gamma \vdash \mathbf{E} : t_1 \Rightarrow t_2}$$

$$\begin{array}{c}
 \text{(FTE-Hole)} \frac{}{\Delta; \Gamma \vdash [\cdot]_{\mathbf{T}} : t \Rightarrow t} \qquad \text{(FTE-Prj)} \frac{\Delta; \Gamma \vdash \mathbf{E} : t \Rightarrow (t_1 \times t_2)}{\Delta; \Gamma \vdash \text{prj}_i \mathbf{E} : t \Rightarrow t_i} \\
 \text{(FTE-Case)} \frac{\Delta; \Gamma \vdash \mathbf{E} : t \Rightarrow (t_1 + t_2) \quad \Delta; \Gamma, x_1 : t_1 \vdash m_1 : t' \quad \Delta; \Gamma, x_2 : t_2 \vdash m_2 : t'}{\Delta; \Gamma \vdash \text{case } \mathbf{E} \text{ of } \text{inj}_1 x_1. m_1 \parallel \text{inj}_2 x_2. m_2 : t \Rightarrow t'} \\
 \text{(FTE-App)} \frac{\Delta; \Gamma \vdash \mathbf{E} : t \Rightarrow (t_1 \rightarrow t_2) \quad \Delta; \Gamma \vdash m_1 : t_1}{\Delta; \Gamma \vdash \mathbf{E} m_1 : t \Rightarrow t_2} \qquad \text{(FTE-Inst)} \frac{\Delta; \Gamma \vdash \mathbf{E} : t_1 \Rightarrow (\forall \alpha :: \kappa. t_2) \quad \Delta \vdash t' :: \kappa}{\Delta; \Gamma \vdash \mathbf{E} [t'] : t_1 \Rightarrow t_2}
 \end{array}$$

Figure 20:  $F_{\omega}$ : Context Typing

$$\boxed{\Gamma \vdash \mathbf{E} : s_1 \Rightarrow s_2}$$

$$\begin{array}{c}
 \text{(DTE-Hole)} \frac{}{\Gamma \vdash [\cdot]_{\mathbf{S}} : s \Rightarrow s} \qquad \text{(DTE-Prj)} \frac{\Gamma \vdash \mathbf{E} : s \Rightarrow (s_1 \times s_2)}{\Gamma \vdash \text{prj}_i \mathbf{E} : s \Rightarrow s_i} \\
 \text{(DTE-Case)} \frac{\Gamma \vdash \mathbf{E} : s \Rightarrow (s_1 + s_2) \quad \Gamma, x_1 : s_1 \vdash e_1 : s' \quad \Gamma, x_2 : s_2 \vdash e_2 : s'}{\Gamma \vdash \text{case } \mathbf{E} \text{ of } \text{inj}_1 x_1. e_1 \parallel \text{inj}_2 x_2. e_2 : s \Rightarrow s'} \\
 \text{(DTE-App)} \frac{\Gamma \vdash \mathbf{E} : s \Rightarrow (s_1 \rightarrow s_2) \quad \Gamma \vdash e_1 : s_1}{\Gamma \vdash \mathbf{E} e_1 : s \Rightarrow s_2} \qquad \text{(DTE-Bind)} \frac{\Gamma \vdash \mathbf{E} : s_1 \Rightarrow (\mathbb{T}_{\ell} s) \quad \Gamma, x : s \vdash e : s_2 \quad \ell \preceq s_2}{\Gamma \vdash \text{bind } x = \mathbf{E} \text{ in } e : s_1 \Rightarrow s_2}
 \end{array}$$

Figure 21: DCC: Context Typing

### 4.2 Back-translation

$$(\mathbb{T}_{\ell} \hat{s})^+ = \forall \beta :: *. ((\hat{\alpha}_{\preceq} \hat{\alpha}_{\ell} \beta) \times (s^{\dagger} \rightarrow \beta)) \rightarrow \beta$$

Figure 22: Back-translation types

*Pf Environment*  $\mathbf{G}_k ::= \cdot \mid \mathbf{G}_k, \mathbf{k} : (\mathbf{s}^\dagger \rightarrow (\mathbb{T}_\ell \mathbf{s})^\dagger)$

Figure 23:  $F_\omega$ : Back Translation Context Grammar

$$\boxed{\hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbf{s}] \stackrel{\text{def}}{=} \delta(\gamma_{\sqsubseteq}(\gamma_{\preceq}(\mathbf{p}\mathbf{f}[\ell \preceq \mathbf{s}])))} \text{ where } \delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$$

$$\gamma_{\sqsubseteq} = \llbracket \mathcal{L}_{\sqsubseteq}^+ \rrbracket$$

$$\gamma_{\preceq} = \llbracket \preceq^+ \rrbracket$$

Figure 24: Back-translation proof constructor

$$\begin{array}{l}
F_\omega \text{ One-layer context } \quad \mathbf{F} ::= \text{prj}_i [\cdot]_T \mid [\cdot]_T \mathbf{m} \mid [\cdot]_T [t] \mid \\
\quad \text{case } [\cdot]_T \text{ of } \text{inj}_1 y. \mathbf{m}_1 \parallel \text{inj}_2 y. \mathbf{m}_2 \\
DCC \text{ One-layer context } \quad \mathbf{F} ::= \text{prj}_i [\cdot]_S \mid [\cdot]_S e \mid \text{bind } x = [\cdot]_S \text{ in } e \mid \\
\quad \text{case } [\cdot]_S \text{ of } \text{inj}_1 y. e_1 \parallel \text{inj}_2 y. e_2
\end{array}$$

$$\boxed{\mathbf{E}^\# = \mathbf{F}_0[\mathbf{F}_1[\dots\mathbf{F}_n]]} \text{ where } \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash \mathbf{F}_0 : \mathbf{t}_1 \Rightarrow \mathbf{s}^\dagger \\
\forall i \in [0, n]. \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash \mathbf{F}_i : \mathbf{t}_{i+1} \Rightarrow \mathbf{t}_i \\
\forall i \in [0, n+1]. \not\exists s_i. \mathbf{t}_i = \mathbf{s}_i^\dagger$$

$$\boxed{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : \mathbf{s}^\dagger \uparrow e} \text{ where } \Gamma^\dagger ::= \cdot \mid \Gamma^\dagger, \mathbf{x} : \mathbf{s}^\dagger \quad \boxed{(\Gamma^\dagger)^\dagger} (\cdot)^\dagger = \cdot \\
\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash \mathbf{m} : \mathbf{s}^\dagger \quad (\Gamma^\dagger, \mathbf{x} : \mathbf{s}^\dagger)^\dagger = (\Gamma^\dagger)^\dagger, \mathbf{x} : \mathbf{s} \\
(\Gamma^\dagger)^\dagger \vdash e : \mathbf{s}$$

$$\begin{array}{l}
(\text{FD-Unit}) \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \langle \rangle : \mathbf{1} \uparrow \langle \rangle \quad (\text{FD-Sum}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : \mathbf{s}_i^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{inj}_i \mathbf{m} : \mathbf{s}_1^\dagger + \mathbf{s}_2^\dagger \uparrow \text{inj}_i e} \\
(\text{FD-Pair}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_1 : \mathbf{s}_1^\dagger \uparrow e_1 \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_2 : \mathbf{s}_2^\dagger \uparrow e_2}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \langle \mathbf{m}_1, \mathbf{m}_2 \rangle : \mathbf{s}_1^\dagger \times \mathbf{s}_2^\dagger \uparrow \langle e_1, e_2 \rangle} \quad (\text{FD-Var}) \quad \frac{(\mathbf{x} : \mathbf{s}^\dagger) \in \Gamma^\dagger}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{x} : \mathbf{s}^\dagger \uparrow \mathbf{x}} \\
(\text{FD-Fun}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger, \mathbf{x} : \mathbf{s}^\dagger \vdash \mathbf{m} : \mathbf{s}_2^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \lambda \mathbf{x} : \mathbf{s}_1^\dagger. \mathbf{m} : \mathbf{s}_1^\dagger \rightarrow \mathbf{s}_2^\dagger \uparrow \lambda \mathbf{x} : \mathbf{s}_1. e} \\
(\text{FD-Return}) \quad \frac{\Sigma; \mathbf{G}_k, \mathbf{k} : (\mathbf{s}^\dagger \rightarrow (\mathbb{T}_\ell \mathbf{s})^\dagger); \Gamma^\dagger \vdash \mathbf{m}[(\mathbb{T}_\ell \mathbf{s})^\dagger / \beta] \langle \text{pf}[\ell \leq \mathbb{T}_\ell \mathbf{s}], \mathbf{k} \rangle : \mathbb{T}_\ell \mathbf{s}^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \Lambda \beta :: * . \mathbf{m} : \mathbb{T}_\ell \mathbf{s}^\dagger \uparrow e} \\
(\text{FD-K}) \quad \frac{\mathbf{k} : (\mathbf{s}^\dagger \rightarrow (\mathbb{T}_\ell \mathbf{s})^\dagger) \in \mathbf{G}_k}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{k} : \mathbf{s}^\dagger \rightarrow \mathbb{T}_\ell \mathbf{s}^\dagger \uparrow \lambda \mathbf{x} : \mathbf{s}. \eta_\ell \mathbf{x}} \\
(\text{FD-Bind}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_p : (\alpha_{\leq} \alpha_\ell \mathbf{s}^\dagger) \times (\mathbf{s}'^\dagger \rightarrow \mathbf{s}^\dagger) \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{prj}_2 \mathbf{m}_p : \mathbf{s}'^\dagger \rightarrow \mathbf{s}^\dagger \uparrow e'}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}[\mathbf{s}^\dagger] \mathbf{m}_p : \mathbf{s}^\dagger \uparrow \text{bind } x = e \text{ in } e' \mathbf{x}} \\
(\text{FD-Subterm}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F} : \mathbf{s}_1^\dagger \Rightarrow \mathbf{s}_2^\dagger \uparrow \mathbf{F}' \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : \mathbf{s}_1^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F}[\mathbf{m}] : \mathbf{s}_2^\dagger \uparrow \mathbf{F}'[e]} \\
(\text{FD-Val}) \quad \frac{\mathbf{E}^\#[\mathbf{u}] \mapsto \mathbf{m}_1 \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_1 : \mathbf{s}_2^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{E}^\#[\mathbf{u}] : \mathbf{s}_2^\dagger \uparrow e} \\
(\text{FD-Stuck}) \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 x_1. \mathbf{m}_1 \parallel \text{inj}_2 x_2. \mathbf{m}_2 : \mathbf{t} \text{ where } \not\exists s'. \mathbf{t} = s'^\dagger \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 y_1. \mathbf{E}^\#[\mathbf{m}_1[y_1/x_1]] \parallel \text{inj}_2 y_2. \mathbf{E}^\#[\mathbf{m}_2[y_2/x_2]] : \mathbf{s}^\dagger \uparrow e \quad (\text{fresh } y'_1, y'_2)}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{E}^\#[\text{case } \mathbf{m} \text{ of } \text{inj}_1 x_1. \mathbf{m}_1 \parallel \text{inj}_2 x_2. \mathbf{m}_2] : \mathbf{s}^\dagger \uparrow e}
\end{array}$$

Figure 25:  $F_\omega$  to DCC Back-translation

$$\boxed{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F} : s_1^\dagger \Rightarrow s_2^\dagger \uparrow F'} \text{ where } \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash \mathbf{F} : s_1^\dagger \Rightarrow s_2^\dagger \\
(\Gamma^\dagger)^\dagger \vdash F' : s_1 \Rightarrow s_2$$

$$\text{(FD-Prj)} \frac{}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{prj}_i [\cdot]_T : (s_1^\dagger \times s_1^\dagger) \Rightarrow s_i^\dagger \uparrow \text{prj}_i [\cdot]_S} \quad \text{(FD-App)} \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash m_2 : s_1^\dagger \uparrow e_2}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash [\cdot]_T m_2 : (s_1^\dagger \rightarrow s_2^\dagger) \Rightarrow s_1^\dagger \uparrow [\cdot]_S e_2}$$

$$\text{(FD-Case)} \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger, \mathbf{x} : s_1^\dagger \vdash m_1 : s_1^\dagger \uparrow e_1 \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger, \mathbf{x} : s_2^\dagger \vdash m_2 : s_2^\dagger \uparrow e_2}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case} [\cdot]_T \text{ of } \text{inj}_1 \mathbf{x}_1. m_1 \parallel \text{inj}_2 \mathbf{x}_2. m_2 : (s_1^\dagger + s_2^\dagger) \Rightarrow s_1^\dagger \uparrow \text{case} [\cdot]_S \text{ of } \text{inj}_1 \mathbf{x}_1. e_1 \parallel \text{inj}_2 \mathbf{x}_2. e_2}$$

Figure 26: Back-translation of  $\mathbf{F}$  contexts

$$\begin{aligned}
\mathbf{W} &::= (\mathbf{G}'_k; \Gamma^{\dagger}) \\
\mathbf{W}_k &\stackrel{\text{def}}{=} \mathbf{G}'_k \mid \mathbf{W} = (\mathbf{G}'_k; \Gamma^{\dagger}) \\
\mathbf{W}_\Gamma &\stackrel{\text{def}}{=} \Gamma^{\dagger} \mid \mathbf{W} = (\mathbf{G}'_k; \Gamma^{\dagger}) \\
\mathbf{W} \supseteq \mathbf{W}' &\stackrel{\text{def}}{=} \mathbf{W}_k \supseteq \mathbf{W}'_k \wedge \mathbf{W}_\Gamma \supseteq \mathbf{W}'_\Gamma \\
\text{Atom}^{\uparrow}[\mathbf{t}]_{\delta}^{\Sigma} &= \{((\mathbf{G}_k; \Gamma^{\dagger}), \mathbf{m}) \mid \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^{\dagger} \vdash \mathbf{m} : \delta(\mathbf{t})\} \\
\text{Rel}^{\uparrow}_{*}^{\Sigma} &= \{(s^{\dagger}, \mathbf{R}) \mid \mathbf{R} \subseteq \text{Atom}^{\uparrow}[\mathbf{s}^{\dagger}]_{\emptyset}^{\Sigma} \wedge \forall (\mathbf{W}, \mathbf{m}) \in \mathbf{R}. \forall \mathbf{W}' \supseteq \mathbf{W}. \\
&\quad (\mathbf{W}', \mathbf{m}) \in \mathbf{R} \wedge \exists e. \Sigma_D; \Sigma_G, \mathbf{W}'_k, \mathbf{W}'_\Gamma \vdash \mathbf{m} : s^{\dagger} \uparrow e\} \\
&\cup \\
&\{(t, \mathbf{R}) \mid \mathbf{R} \subseteq \text{Atom}^{\uparrow}[\mathbf{t}]_{\emptyset}^{\Sigma} \wedge \nexists s. t = s^{\dagger} \wedge \\
&\quad \forall (\mathbf{W}, \mathbf{m}) \in \mathbf{R}, \mathbf{W}' \supseteq \mathbf{W}. (\mathbf{W}', \mathbf{m}) \in \mathbf{R} \wedge \\
&\quad \forall \mathbf{E}^{\#}, t. \\
&\quad (\Sigma_D; \Sigma_G, \mathbf{W}_k, \mathbf{W}_\Gamma \vdash \mathbf{E}^{\#} : t \Rightarrow s^{\dagger} \wedge \\
&\quad (\forall \mathbf{W}', \mathbf{u}. (\mathbf{W}' \supseteq \mathbf{W} \wedge \Sigma_D; \Sigma_G, \mathbf{W}'_k, \mathbf{W}'_\Gamma \vdash \mathbf{u} : t) \Rightarrow \exists e. \Sigma; \mathbf{W}'_k; \mathbf{W}'_\Gamma \vdash \mathbf{E}^{\#}[\mathbf{u}] : s^{\dagger} \uparrow e) \Rightarrow \\
&\quad \exists e'. \Sigma; \mathbf{W}_k; \mathbf{W}_\Gamma \vdash \mathbf{E}^{\#}[\mathbf{m}] : s^{\dagger} \uparrow e'\} \\
\text{Rel}^{\uparrow}_{\kappa_1 \rightarrow \kappa_2}^{\Sigma} &= \{(t_1, \mathbf{R}_1) \mid \\
&\quad \forall (t_2, \mathbf{R}_2) \in \text{Rel}^{\uparrow}_{\kappa_1}^{\Sigma}. ((t_1 t_2), \mathbf{R}_1 (t_2, \mathbf{R}_2)) \in \text{Rel}^{\uparrow}_{\kappa_2}^{\Sigma} \wedge \\
&\quad \forall (t'_2, \mathbf{R}'_2) \in \text{Rel}^{\uparrow}_{\kappa_1}^{\Sigma}. \\
&\quad (t_2, \mathbf{R}_2) \equiv_{\kappa_1}^{\Sigma} (t'_2, \mathbf{R}'_2) \Rightarrow (\mathbf{R}_1 (t_2, \mathbf{R}_2)) \equiv_{\kappa_2}^{\Sigma} (\mathbf{R}_1 (t'_2, \mathbf{R}'_2))\} \\
\delta \equiv^{\Sigma} \delta' &\stackrel{\text{def}}{=} \forall \alpha :: \kappa \in \text{dom}(\delta). \delta(\alpha) \equiv_{\kappa}^{\Sigma} \delta'(\alpha) \\
(t_1, \mathbf{R}_1) \equiv_{\kappa}^{\Sigma} (t_2, \mathbf{R}_2) &\stackrel{\text{def}}{=} t_1 \equiv t_2 \wedge \mathbf{R}_1 \equiv_{\kappa}^{\Sigma} \mathbf{R}_2 \\
\mathbf{R}_1 \equiv_{*}^{\Sigma} \mathbf{R}_2 &\stackrel{\text{def}}{=} \forall \mathbf{W}, \mathbf{m}_1. (\mathbf{W}, \mathbf{m}_1) \in \mathbf{R}_1 \iff (\mathbf{W}, \mathbf{m}_1) \in \mathbf{R}_2 \\
\mathbf{R}_1 \equiv_{\kappa_1 \rightarrow \kappa_2}^{\Sigma} \mathbf{R}_2 &\stackrel{\text{def}}{=} \forall (t', \mathbf{R}') \in \text{Rel}^{\uparrow}_{\kappa_1}^{\Sigma}. (\mathbf{R}_1 (t', \mathbf{R}')) \equiv_{\kappa_2}^{\Sigma} (\mathbf{R}_2 (t', \mathbf{R}')) \\
\mathcal{T}^{\uparrow}[\mathbf{t} :: *]_{\delta}^{\Sigma} &= \mathcal{E}^{\uparrow}[\mathbf{t}]_{\delta}^{\Sigma} \\
\mathcal{T}^{\uparrow}[\alpha :: \kappa_1 \rightarrow \kappa_2]_{\delta}^{\Sigma} &= \delta_{\mathbf{R}}(\alpha) \\
\mathcal{T}^{\uparrow}[\lambda \alpha :: \kappa_1. t :: \kappa_1 \rightarrow \kappa_2]_{\delta}^{\Sigma} &= \lambda_{\mathbf{R}}(t_1, \mathbf{R}_1). \{\mathcal{T}^{\uparrow}[\mathbf{t} :: \kappa_2]_{\delta}^{\Sigma}[\alpha :: \kappa_1 \mapsto (t_1, \mathbf{R}_1)]\} \\
\mathcal{T}^{\uparrow}[\mathbf{t}_1 t_2 :: \kappa_2]_{\delta}^{\Sigma} &= (\mathcal{T}^{\uparrow}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\delta}^{\Sigma} \\
&\quad (\delta(t_2), \mathcal{T}^{\uparrow}[\mathbf{t}_2 :: \kappa_1]_{\delta}^{\Sigma}))
\end{aligned}$$

Figure 27: Back-translation: Logical Relations (Higher-Order)

$$\begin{aligned}
Atom^{\uparrow \text{val}}[t]_{\delta}^{\Sigma} &= \{ ((\mathbf{G}_k; \Gamma^{\dagger}), \mathbf{u}) \mid \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^{\dagger} \vdash \mathbf{u} : \delta(t) \} \\
Atom^{\uparrow \text{ctx}}[t, s^{\dagger}]_{\delta}^{\Sigma} &= \{ ((\mathbf{G}_k; \Gamma^{\dagger}), \mathbf{E}^{\#}) \mid \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^{\dagger} \vdash \mathbf{E}^{\#} : \delta(t) \Rightarrow s^{\dagger} \} \\
\mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \mathbf{m}) \mid \exists e. (\mathbf{W}, \mathbf{m}) \in Atom^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} \wedge \Sigma; \mathbf{W}_k; \mathbf{W}_{\Gamma} \vdash \mathbf{m} : s^{\dagger} \uparrow e \} \\
\mathcal{E}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} &= \mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} \\
\mathcal{V}^{\uparrow}[\alpha]_{\delta}^{\Sigma} &= \delta_{\mathbf{R}}(\alpha) \\
\mathcal{V}^{\uparrow}[t \times t']_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \langle \mathbf{m}_1, \mathbf{m}_2 \rangle) \in Atom^{\uparrow \text{val}}[t \times t']_{\delta}^{\Sigma} \mid (\mathbf{W}, \mathbf{m}_1) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma} \wedge (\mathbf{W}, \mathbf{m}_2) \in \mathcal{E}^{\uparrow}[t']_{\delta}^{\Sigma} \} \\
\mathcal{V}^{\uparrow}[t + t']_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \text{inj}_1 \mathbf{m}) \in Atom^{\uparrow \text{val}}[t + t']_{\delta}^{\Sigma} \mid (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma} \} \\
&\cup \{ (\mathbf{W}, \text{inj}_2 \mathbf{m}) \in Atom^{\uparrow \text{val}}[t + t']_{\delta}^{\Sigma} \mid (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\uparrow}[t']_{\delta}^{\Sigma} \} \\
\mathcal{V}^{\uparrow}[t' \rightarrow t]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \lambda x: t'. \mathbf{m}) \in Atom^{\uparrow \text{val}}[t' \rightarrow t]_{\delta}^{\Sigma} \mid \\
&\quad \forall \mathbf{W}' \supseteq \mathbf{W}, (\mathbf{W}', \mathbf{m}') \in \mathcal{E}^{\uparrow}[t']_{\delta}^{\Sigma}. (\mathbf{W}', \mathbf{m}[\mathbf{m}'/x]) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma} \} \\
\mathcal{V}^{\uparrow}[\forall \alpha :: \kappa. t]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \Lambda \alpha :: \kappa. \mathbf{m}) \in Atom^{\uparrow \text{val}}[\forall \alpha :: \kappa. t]_{\delta}^{\Sigma} \mid \\
&\quad \forall \mathbf{W}' \supseteq \mathbf{W}, (\mathbf{t}', \mathbf{R}) \in Rel^{\uparrow \Sigma}_{\kappa}. (\mathbf{W}', \mathbf{m}[\mathbf{t}'/\alpha]) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma}[\alpha \mapsto (\mathbf{t}', \mathbf{R})] \} \\
\mathcal{V}^{\uparrow}[t \ t']_{\delta}^{\Sigma} &= (\mathcal{T}^{\uparrow}[t :: \kappa' \rightarrow *]_{\delta}^{\Sigma} (\delta_1(t'), \mathcal{T}^{\uparrow}[t' :: \kappa']_{\delta}^{\Sigma})) \\
\mathcal{E}^{\uparrow}[\alpha]_{\delta}^{\Sigma} &= \delta_{\mathbf{R}}(\alpha) \\
\mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \mathbf{m}) \in Atom^{\uparrow}[t]_{\delta}^{\Sigma} \mid \\
&\quad \forall \mathbf{W}' \supseteq \mathbf{W}, (\mathbf{W}', \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[t, s^{\dagger}]_{\delta}^{\Sigma}. (\mathbf{W}', \mathbf{E}^{\#}[\mathbf{m}]) \in \mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} \} \\
\mathcal{K}^{\uparrow}[t, s^{\dagger}]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \mathbf{E}^{\#}) \in Atom^{\uparrow \text{ctx}}[t, s^{\dagger}]_{\delta}^{\Sigma} \mid \\
&\quad \forall \mathbf{W}' \supseteq \mathbf{W}, (\mathbf{W}', \mathbf{m}) \in \mathcal{V}^{\uparrow}[t]_{\delta}^{\Sigma}. (\mathbf{W}', \mathbf{E}^{\#}[\mathbf{m}]) \in \mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma} \} \\
\mathcal{G}^{\uparrow}[\cdot]_{\delta}^{\Sigma} &= \{ \emptyset \} \\
\mathcal{G}^{\uparrow}[\Gamma, x : t]_{\delta}^{\Sigma} &= \{ (\mathbf{W}, \gamma[x \mapsto \mathbf{m}]) \mid (\mathbf{W}, \gamma) \in \mathcal{G}^{\uparrow}[\Gamma]_{\delta}^{\Sigma} \wedge (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\Sigma} \} \\
\mathcal{D}^{\uparrow}[\cdot]_{\delta}^{\Sigma} &= \{ \emptyset \} \\
\mathcal{D}^{\uparrow}[\Delta, \alpha :: \kappa]_{\delta}^{\Sigma} &= \{ \delta[\alpha \mapsto (t, \mathbf{R})] \mid \delta \in \mathcal{D}^{\uparrow}[\Delta]_{\delta}^{\Sigma} \wedge (t, \mathbf{R}) \in Rel^{\uparrow \Sigma}_{\kappa} \wedge \Sigma_D \vdash t :: \kappa \} \\
\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_k, \Gamma^{\dagger}, \Gamma \models^{\uparrow} \mathbf{m} : t &= \forall \Gamma_1^{\dagger}, \Gamma_2^{\dagger}, \delta, \gamma, \gamma. \\
&\quad \Gamma_1^{\dagger} \uplus \Gamma_2^{\dagger} = \Gamma^{\dagger} \wedge \delta \in \mathcal{D}^{\uparrow}[\Delta]_{\delta}^{\mathbf{D}_{\ell}} \wedge \\
&\quad ((\mathbf{G}_k; \Gamma_2^{\dagger}), \gamma) \in \mathcal{G}^{\uparrow}[\Gamma_1^{\dagger}]_{\delta}^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}} \wedge \\
&\quad ((\mathbf{G}_k; \Gamma_2^{\dagger}), \gamma) \in \mathcal{G}^{\uparrow}[\Gamma]_{\delta}^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}} \implies \\
&\quad ((\mathbf{G}_k; \Gamma_2^{\dagger}), \delta(\gamma(\gamma(\mathbf{m})))) \in \mathcal{E}^{\uparrow}[t]_{\delta}^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}}
\end{aligned}$$

Figure 28: Back-translation: Logical Relations

## 5 Theorems

### 5.1 Noninterference

#### Theorem 5.1 (Noninterference)

If  $\Gamma \vdash e : s$  then  $\Gamma \vdash e \approx_{\zeta} e : s$

#### Proof

By induction on  $\Gamma \vdash e : s$ . The case for bind is the only interesting case, which has two cases: if  $\ell \sqsubseteq \zeta$ , then the result follows by induction, otherwise the result follows from Lemma 5.2.  $\square$

#### Lemma 5.2

If  $\ell \preceq s$  and  $\ell \not\sqsubseteq \zeta$  then  $\forall (e_1, e_2) \in \text{Atom}[s]. (e_1, e_2) \in \mathcal{E}[\![s]\!]_{\zeta}$

#### Proof

By induction on  $\ell \preceq s$ . There are two interesting cases; the others follow easily by induction.

Show for arbitrary  $e_1, e_2$ ,  $(e_1, e_2) \in \text{Atom}[\![T_{\ell'} s]\!]$  implies  $(e_1, e_2) \in \mathcal{E}[\![T_{\ell'} s_1]\!]_{\zeta}$ .

Since our language is terminating, we know  $e_1 \mapsto^* \eta_{\ell'} e'_1$  and  $e_2 \mapsto^* \eta_{\ell'} e'_2$ , so it suffices to show  $\ell' \sqsubseteq \zeta \implies (e'_1, e'_2) \in \mathcal{E}[\![s_1]\!]_{\zeta}$ .

#### Case P-Label1

We must show  $(e'_1, e'_2) \in \mathcal{E}[\![s_1]\!]_{\zeta}$ . By assumption,  $\ell \not\sqsubseteq \zeta$ , and by P-Label1, we know  $\ell \preceq s_1$ . By induction we know  $\forall (e_1, e_2) \in \text{Atom}[s_1]. (e_1, e_2) \in \mathcal{E}[\![s_1]\!]_{\zeta}$ .

#### Case P-Label2

By assumption  $\ell \not\sqsubseteq \zeta$ , therefore  $\ell' \not\sqsubseteq \zeta$ . This immediately implies that  $(e_1, e_2) \in \mathcal{V}[\![T_{\ell'} s]\!]_{\zeta}$ .  $\square$

### 5.2 Transitivity and Symmetry

For technical reasons, we need to know the target relation is symmetric and transitive. Normally this is proved by showing the logical relation respects contextual equivalence. However, leaving the relation open with respect to  $\mathbf{D}; \mathbf{G}$  makes proving this difficult or impossible.

Instead, we define functions  $\text{sym}()$  and  $\text{trans}()$  that compute symmetric and transitive versions of given relations, then extend these functions to  $\rho$ s. We then prove the symmetric-transitive result of  $[\![\mathcal{L}_{\ell}^+]\!]_{\zeta}^{\Sigma}$  is itself  $[\![\mathcal{L}_{\ell}^+]\!]_{\zeta}^{\Sigma}$ , yielding restricted, but strong enough, symmetry and transitivity lemmas.

#### 5.2.1 Target Symmetry

$$\begin{aligned}
 \text{sym}(\mathbf{R} :: *) &= \{ (m_1, m_2) \mid (m_2, m_1) \in \mathbf{R} \} \\
 \text{sym}(\mathbf{R} :: \kappa_1 \rightarrow \kappa_2) &= \lambda_R (t_1, t_2, \mathbf{R}') . \text{sym}(\mathbf{R}(t_2, t_1, \text{sym}(\mathbf{R}')))) \\
 \text{sym}(\emptyset) &= \emptyset \\
 \text{sym}(\rho[\alpha \mapsto (t_1, t_2, \mathbf{R})]) &= \text{sym}(\rho)[\alpha \mapsto (t_2, t_1, \text{sym}(\mathbf{R}))]
 \end{aligned}$$

Figure 29:  $F_{\omega}$ : Symmetric Relations

#### Lemma 5.3 (sym Identity)

$\text{sym}(\text{sym}(\mathbf{R})) = \mathbf{R}$ .

**Lemma 5.4 (General Target Relation Symmetry)**

If  $\Delta \vdash t :: \kappa$  and  $\rho \in \mathcal{D}[\Delta]^{\mathbf{D};\mathbf{G}}$  then  $\text{sym}(\mathcal{T}[t :: \kappa]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \mathcal{T}[t :: \kappa]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$

**Proof**

By induction on the derivation  $\Delta \vdash t :: \kappa$ .

**Case** (FK-Unit)  $\frac{}{\Delta \vdash 1 :: *}$

We must show  $\text{sym}(\mathcal{T}[1 :: *]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{T}[1 :: *]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

It suffices to show  $\text{sym}(\mathcal{V}[1]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[1]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

By definition, it suffices to show, for all  $(\mathbf{m}_2, \mathbf{m}_1) \in \mathcal{V}[1]_{\rho}^{\mathbf{D};\mathbf{G}}$ , that  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[1]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ , which follows by definition of  $\mathcal{V}[1]_{\rho}^{\mathbf{D};\mathbf{G}}$ .

**Case** (FK-Var)  $\frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa}$

We must show  $\text{sym}(\mathcal{T}[\alpha :: \kappa]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\alpha :: \kappa]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

By definition it suffices to show  $\text{sym}(\rho(\alpha)) = \text{sym}(\rho)(\alpha)$ , which follows by definition of  $\text{sym}()$ .

**Case** (FK-Pair)  $\frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \times t_2 :: *}$

Follows from the IH.

**Case** (FK-Sum)  $\frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *}$

Follows from the IH.

**Case** (FK-Arrow)  $\frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \rightarrow t_2 :: *}$

It suffices to show  $\text{sym}(\mathcal{V}[t_1 \rightarrow t_2]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[t_1 \rightarrow t_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

We show only one direction, since the other is exactly symmetric.

For arbitrary  $(\lambda x:t_1.\mathbf{m}_2, \lambda x:t_1.\mathbf{m}_1) \in \mathcal{V}[t_1 \rightarrow t_2]_{\rho}^{\mathbf{D};\mathbf{G}}$ , we must show  $(\lambda x:t_1.\mathbf{m}_1, \lambda x:t_1.\mathbf{m}_2) \in \mathcal{V}[t_1 \rightarrow t_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $(\mathbf{m}, \mathbf{m}') \in \mathcal{E}[t_1]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ , that  $(\mathbf{m}_1[x/\mathbf{m}], \mathbf{m}_2[x/\mathbf{m}']) \in \mathcal{E}[t_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

By the IH applied to  $\Delta \vdash t_2 :: *$ , it suffices to show  $(\mathbf{m}_2[x/\mathbf{m}'], \mathbf{m}_1[x/\mathbf{m}]) \in \mathcal{E}[t_2]_{\rho}^{\mathbf{D};\mathbf{G}}$ .

By the IH applied to  $\Delta \vdash t_1 :: *$ , we know  $\text{sym}(\mathcal{E}[t_1]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{E}[t_1]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$  and therefore  $(\mathbf{m}', \mathbf{m}) \in \mathcal{E}[t_1]_{\rho}^{\mathbf{D};\mathbf{G}}$ .

Instantiating  $(\lambda x:t_1.\mathbf{m}_2, \lambda x:t_1.\mathbf{m}_1) \in \mathcal{V}[t_1 \rightarrow t_2]_{\rho}^{\mathbf{D};\mathbf{G}}$  with  $(\mathbf{m}', \mathbf{m})$ , we conclude  $(\mathbf{m}_2[x/\mathbf{m}'], \mathbf{m}_1[x/\mathbf{m}]) \in \mathcal{E}[t_2]_{\rho}^{\mathbf{D};\mathbf{G}}$ .

**Case** (FK-Abs)  $\frac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha :: \kappa. t :: *}$

It suffices to show  $\text{sym}(\mathcal{V}[\forall \alpha :: \kappa. t]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[\forall \alpha :: \kappa. t]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

We show only one direction, since the other is exactly symmetric.

For arbitrary  $(\Lambda \alpha :: \kappa.\mathbf{m}_2, \Lambda \alpha :: \kappa.\mathbf{m}_1) \in \mathcal{V}[\forall \alpha :: \kappa. t]_{\rho}^{\mathbf{D};\mathbf{G}}$ , we must show  $(\Lambda \alpha :: \kappa.\mathbf{m}_1, \Lambda \alpha :: \kappa.\mathbf{m}_2) \in \mathcal{V}[\forall \alpha :: \kappa. t]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $\pi \in \text{Rel}_{\kappa}^{\mathbf{D};\mathbf{G}}$ , we must show  $(\mathbf{m}_1[\alpha/\pi_1], \mathbf{m}_2[\alpha/\pi_2]) \in \mathcal{E}[t]_{\text{sym}(\rho)[\alpha \mapsto \pi]}^{\mathbf{D};\mathbf{G}}$ .

Let  $\pi' = (\pi_2, \pi_1, \text{sym}(\pi_R))$ .



Instantiating  $(\Lambda\alpha::\kappa.m_2, \Lambda\alpha::\kappa.m_1) \in \mathcal{V}[\forall\alpha::\kappa. t]_{\rho}^{\mathbf{D};\mathbf{G}}$  with  $\pi'$  we know  $(m_2[\alpha/\pi_2], m_1[\alpha/\pi_1]) \in \mathcal{E}[\mathbf{t}]_{\rho[\alpha \mapsto \pi']}^{\mathbf{D};\mathbf{G}}$ .

Therefore, it suffices to show  $\text{sym}(\mathcal{E}[\mathbf{t}]_{\rho[\alpha \mapsto \pi']}^{\mathbf{D};\mathbf{G}}) \equiv_*^{\mathbf{D};\mathbf{G}} \mathcal{E}[\mathbf{t}]_{\text{sym}(\rho)[\alpha \mapsto \pi]}^{\mathbf{D};\mathbf{G}}$ , which follows from the IH applied to  $\Delta, \alpha :: \kappa \vdash \mathbf{t} :: *$ , with  $\rho[\alpha \mapsto (\pi_2, \pi_1, \text{sym}(\pi_R))]$ .

$$\text{Case (FK-Fun)} \frac{\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2}{\Delta \vdash \lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2}$$

We must show  $\text{sym}(\mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

By definition, it suffices to show

$$\lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}').\text{sym}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} \lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}').\mathcal{T}[\mathbf{t} :: \kappa_2]_{\text{sym}(\rho)[\alpha \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}')] }^{\mathbf{D};\mathbf{G}}$$

For arbitrary  $\pi \in \text{Rel}_{\kappa}^{\mathbf{D};\mathbf{G}}$ , it suffices to show

$$\text{sym}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho[\alpha \mapsto (\pi_2, \pi_1, \text{sym}(\pi_R))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t} :: \kappa_2]_{\text{sym}(\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}$$

By the IH applied to  $\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$ , we know

$$\text{sym}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t} :: \kappa_2]_{\text{sym}(\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}$$

By definition and Lemma 5.3, we know  $\text{sym}(\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]) = \text{sym}(\rho)[\alpha \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}')]$ , therefore

$$\text{sym}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho[\alpha \mapsto (\pi_2, \pi_1, \text{sym}(\pi_R))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t} :: \kappa_2]_{\text{sym}(\rho[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \text{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}$$

$$\text{Case (FK-App)} \frac{\Delta \vdash \mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2 \quad \Delta \vdash \mathbf{t}_2 :: \kappa_1}{\Delta \vdash \mathbf{t}_1 \mathbf{t}_2 :: \kappa_2}$$

We must show  $\text{sym}(\mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

Let  $\pi = (\rho_1 \mathbf{t}_2, \rho_2 \mathbf{t}_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_1]_{\rho}^{\mathbf{D};\mathbf{G}})$  and  $\pi' = (\rho_2 \mathbf{t}_2, \rho_1 \mathbf{t}_2, \text{sym}(\mathcal{T}[\mathbf{t}_2 :: \kappa_1]_{\rho}^{\mathbf{D};\mathbf{G}}))$ .

By definition of  $\mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}$ , it suffices to show

$$\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}} (\pi_1, \pi_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_1]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}).$$

By the IH applied to  $\Delta \vdash \mathbf{t}_2 :: \kappa_1$ , it suffices to show  $\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}} \pi'$ .

By the IH applied to  $\Delta \vdash \mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2$  we know

$$\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}.$$

By definition of  $\equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}}$ , we know

$$\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}) \pi' \equiv_{\kappa_1 \rightarrow \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}} \pi'.$$

By definition, we know  $\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}) = \lambda_R(\mathbf{t}'_1, \mathbf{t}'_2, \mathbf{R}').\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}(\mathbf{t}'_2, \mathbf{t}'_1, \text{sym}(\mathbf{R}')))$

Therefore,  $\text{sym}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}} \pi'$ .

□

### Lemma 5.5 (Lattice Interp. Self-Symmetric)

Let  $\rho = \llbracket \mathcal{L}_{\ell}^+ \rrbracket_{\zeta}^{\Sigma}$ .  $\text{sym}(\rho) = \rho$

### Lemma 5.6 (Target Expression Relation Symmetry)

If  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}$  then  $(\mathbf{m}_2, \mathbf{m}_1) \in \mathcal{E}[\mathbf{t}]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$

#### Proof

We must show  $(\mathbf{m}_2, \mathbf{m}_1) \in \mathcal{E}[\mathbf{t}]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$

By Lemma 5.7 (General Target Relation Symmetry), we know  $\text{sym}(\mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_*^{\mathbf{D};\mathbf{G}} \mathcal{E}[\mathbf{t}]_{\text{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$ .

By definition of  $\equiv_*^{\mathbf{D};\mathbf{G}}$ , it suffices to show  $(\mathbf{m}_2, \mathbf{m}_1) \in \text{sym}(\mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}})$

By definition of  $\text{sym}(\mathbf{R} :: *)$ ,  $(\mathbf{m}_2, \mathbf{m}_1) \in \text{sym}(\mathbf{R})$  if  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R}$ .

Take  $\mathbf{R}$  to be  $\mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}$

It suffices to show  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}$ , which follows by assumption.  $\square$

### Lemma 5.7 (Target Relation Symmetric under Lattice Interp.)

Let  $\rho = \llbracket \mathcal{L}_{\ell}^+ \rrbracket_{\zeta}^{\Sigma}$ .

If  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$  then  $(\mathbf{m}_2, \mathbf{m}_1) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$ .

### 5.2.2 Target Transitivity

$$\begin{aligned}
\text{fst}_{\mathbf{R}}(\mathbf{R})_* &= \{(\mathbf{m}_1, \langle \rangle) \mid (\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R}\} \\
\text{fst}_{\mathbf{R}}(\mathbf{R})_{\kappa_1 \rightarrow \kappa_2} &= \lambda_{\mathbf{R}} \pi. \text{fst}_{\mathbf{R}}(\mathbf{R} \pi)_{\kappa_2} \\
\text{snd}_{\mathbf{R}}(\mathbf{R})_* &= \{(\langle \rangle, \mathbf{m}_2) \mid (\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R}\} \\
\text{snd}_{\mathbf{R}}(\mathbf{R})_{\kappa_1 \rightarrow \kappa_2} &= \lambda_{\mathbf{R}} \pi. \text{snd}_{\mathbf{R}}(\mathbf{R} \pi)_{\kappa_2} \\
\mathbf{1}_* &= \mathbf{1} \\
\mathbf{1}_{\kappa_1 \rightarrow \kappa_2} &= \lambda_{\alpha :: \kappa_1}. \mathbf{1}_{\kappa_2} \\
\text{trans}(\mathbf{R}_1, \mathbf{R}_2)_* &= \{(\mathbf{m}_1, \mathbf{m}_3) \mid (\mathbf{m}_1, \mathbf{m}_2) \in \mathbf{R}_1 \wedge (\mathbf{m}_2, \mathbf{m}_3) \in \mathbf{R}_2\} \\
\text{trans}(\mathbf{R}_1, \mathbf{R}_2)_{\kappa_1 \rightarrow \kappa_2} &= \lambda_{\mathbf{R}}(\mathbf{t}_1, \mathbf{t}_3, \mathbf{R}). \text{trans}((\mathbf{R}_1(\mathbf{t}_1, \mathbf{1}_{\kappa_1}, \text{fst}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})), (\mathbf{R}_2(\mathbf{1}_{\kappa_1}, \mathbf{t}_2, \text{snd}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})))_{\kappa_2} \\
\text{trans}(\emptyset, \emptyset) &= \emptyset \\
\text{trans}(\rho[\alpha :: \kappa \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_1)], \rho'[\alpha :: \kappa \mapsto (\mathbf{t}_2, \mathbf{t}_3, \mathbf{R}_2)]) &= (\text{trans}(\rho, \rho'))[\alpha \mapsto (\mathbf{t}_1, \mathbf{t}_3, \text{trans}(\mathbf{R}_1, \mathbf{R}_2)_{\kappa})]
\end{aligned}$$

Figure 30:  $F_{\omega}$ : Transitive Relations

### Lemma 5.8 (trans Identity)

$$\text{trans}(\text{fst}_{\mathbf{R}}(\mathbf{R})_{\kappa}, \text{snd}_{\mathbf{R}}(\mathbf{R})_{\kappa})_{\kappa} = \mathbf{R}$$

### Lemma 5.9 (fst Identity)

$$\text{fst}_{\mathbf{R}}(\text{trans}(\mathbf{R}_1, \mathbf{R}_2)_{\kappa})_{\kappa} = \mathbf{R}_1$$

### Lemma 5.10 (snd Identity)

$$\text{snd}_{\mathbf{R}}(\text{trans}(\mathbf{R}_1, \mathbf{R}_2)_{\kappa})_{\kappa} = \mathbf{R}_2$$

### Lemma 5.11 (General Target Relation Transitivity)

If  $\Delta \vdash \mathbf{t} :: \kappa$ ,  $\rho \in \mathcal{D}[\Delta]^{\mathbf{D};\mathbf{G}}$ ,  $\rho' \in \mathcal{D}[\Delta]^{\mathbf{D};\mathbf{G}}$ , and  $\rho_2 = \rho'_1$ , then

$$\text{trans}(\mathcal{T}[\mathbf{t} :: \kappa]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{T}[\mathbf{t} :: \kappa]_{\rho'}^{\mathbf{D};\mathbf{G}})_{\kappa} \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t} :: \kappa]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}.$$

### Proof

By induction on the derivation  $\Delta \vdash \mathbf{t} :: \kappa$ .

Case (FK-Unit)  $\frac{}{\Delta \vdash \mathbf{1} :: *}$

It suffices to show  $\text{trans}(\mathcal{V}[\mathbf{1}]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\mathbf{1}]_{\rho'}^{\mathbf{D};\mathbf{G}})_* \equiv_*^{\mathbf{D};\mathbf{G}} \mathcal{V}[\mathbf{1}]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $(\mathbf{m}_1, \mathbf{m}_3) \in \text{trans}(\mathcal{V}[\mathbf{1}]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\mathbf{1}]_{\rho'}^{\mathbf{D};\mathbf{G}})_*$ , we must show  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{V}[\mathbf{1}]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

By definition,  $(\mathbf{m}_1, \mathbf{m}_3) \in \text{trans}(\mathcal{V}[\mathbf{1}]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\mathbf{1}]_{\rho'}^{\mathbf{D};\mathbf{G}})_*$  implies  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[\mathbf{1}]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{V}[\mathbf{1}]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Therefore  $(\mathbf{m}_1, \mathbf{m}_3) = (\langle \rangle, \langle \rangle)$ , thus  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{V}[\mathbf{1}]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

$$\text{Case (FK-Var)} \frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa}$$

It suffices to show  $\text{trans}(\rho_{\mathbf{R}}(\alpha), \rho'_{\mathbf{R}}(\alpha))_{\kappa} \equiv_{\kappa}^{\mathbf{D};\mathbf{G}} \text{trans}(\rho, \rho')_{\kappa} \mathbf{R}(\alpha)$ , which follows by definition of  $\text{trans}(\rho, \rho')_{\kappa}$ .

$$\text{Case (FK-Pair)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \times t_2 :: *}$$

It suffices to show  $\text{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*} \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[\![t_1 \times t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $(\mathbf{m}_1, \mathbf{m}_3) \in \text{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ , we must show  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{V}[\![t_1 \times t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

By definition of  $\text{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ , we know  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[\![t_1 \times t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Therefore,  $\mathbf{m}_1 = \langle \mathbf{m}'_1, \mathbf{m}''_1 \rangle$ ,  $\mathbf{m}_2 = \langle \mathbf{m}'_2, \mathbf{m}''_2 \rangle$ , and  $\mathbf{m}_3 = \langle \mathbf{m}'_3, \mathbf{m}''_3 \rangle$ .

So it suffices to show  $(\mathbf{m}'_1, \mathbf{m}'_3) \in \mathcal{E}[\![t_1]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}''_1, \mathbf{m}''_3) \in \mathcal{E}[\![t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

We know  $(\mathbf{m}'_1, \mathbf{m}'_2) \in \mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}'_2, \mathbf{m}'_3) \in \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}}$ . By definition, therefore,  $(\mathbf{m}'_1, \mathbf{m}'_3) \in \text{trans}(\mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ .

Similarly we know  $(\mathbf{m}''_1, \mathbf{m}''_3) \in \text{trans}(\mathcal{E}[\![t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ .

By IH applied to  $\Delta \vdash t_1 :: *$ , we know  $(\mathbf{m}'_1, \mathbf{m}'_3) \in \mathcal{E}[\![t_1]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

By IH applied to  $\Delta \vdash t_2 :: *$ , we know  $(\mathbf{m}''_1, \mathbf{m}''_3) \in \mathcal{E}[\![t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

$$\text{Case (FK-Sum)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *}$$

Follows by IH, similar to the previous case.

$$\text{Case (FK-Arrow)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \rightarrow t_2 :: *}$$

It suffices to show  $\text{trans}(\mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*} \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $(\mathbf{m}_1, \mathbf{m}_3) \in \text{trans}(\mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ , we must show  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

By definition of  $\text{trans}(\mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ , we know  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{V}[\![t_1 \rightarrow t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Therefore,  $\mathbf{m}_1 = \lambda x:t_1. \mathbf{m}'_1$ ,  $\mathbf{m}_2 = \lambda x:t_1. \mathbf{m}'_2$ , and  $\mathbf{m}_3 = \lambda x:t_1. \mathbf{m}'_3$ .

For arbitrary  $(\mathbf{m}, \mathbf{m}'') \in \mathcal{E}[\![t_1]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ , it suffices to show  $(\mathbf{m}'_1[\mathbf{m}/x], \mathbf{m}'_3[\mathbf{m}''/x]) \in \mathcal{E}[\![t_2]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

By IH applied to  $\Delta \vdash t_2 :: *$ , it suffices to show  $(\mathbf{m}'_1[\mathbf{m}/x], \mathbf{m}'_3[\mathbf{m}''/x]) \in \text{trans}(\mathcal{E}[\![t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ .

By IH applied to  $\Delta \vdash t_1 :: *$ , we know  $\text{trans}(\mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*} \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{E}[\![t_1]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

Therefore,  $(\mathbf{m}, \mathbf{m}'') \in \text{trans}(\mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ .

By definition of  $\text{trans}(\mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ ,  $(\mathbf{m}, \mathbf{m}') \in \mathcal{E}[\![t_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}', \mathbf{m}'') \in \mathcal{E}[\![t_1]\!]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Therefore,  $(\mathbf{m}'_1[\mathbf{m}/x], \mathbf{m}'_2[\mathbf{m}'/x]) \in \mathcal{V}[\![t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$  and  $(\mathbf{m}'_2[\mathbf{m}'/x], \mathbf{m}'_3[\mathbf{m}''/x]) \in \mathcal{V}[\![t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Therefore, by definition,  $(\mathbf{m}'_1[\mathbf{m}/x], \mathbf{m}'_3[\mathbf{m}''/x]) \in \text{trans}(\mathcal{E}[\![t_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\![t_2]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*}$ .

$$\text{Case (FK-Abs)} \frac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha :: \kappa. t :: *}$$

It suffices to show  $\text{trans}(\mathcal{V}[\![\forall \alpha :: \kappa. t]\!]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{V}[\![\forall \alpha :: \kappa. t]\!]_{\rho'}^{\mathbf{D};\mathbf{G}})_{*} \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{V}[\![\forall \alpha :: \kappa. t]\!]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

For arbitrary  $(\mathbf{m}_1, \mathbf{m}_3) \in \text{trans}(\mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho}^{\text{D;G}}, \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho'}^{\text{D;G}})_*$ , we must show  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\text{trans}(\rho, \rho')}^{\text{D;G}}$ .

By definition of  $\text{trans}(\mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho}^{\text{D;G}}, \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho'}^{\text{D;G}})_*$ , we know  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho}^{\text{D;G}}$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho'}^{\text{D;G}}$ .

Therefore,  $\mathbf{m}_1 = \Lambda\alpha::\kappa. \mathbf{m}'_1$ ,  $\mathbf{m}_2 = \Lambda\alpha::\kappa. \mathbf{m}'_2$ , and  $\mathbf{m}_3 = \Lambda\alpha::\kappa. \mathbf{m}'_3$ .

For arbitrary  $\pi \in \text{Rel}_{\kappa}^{\text{D;G}}$ , we must show  $(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\mathbf{t}]_{\text{trans}(\rho, \rho')[\alpha::\kappa \mapsto \pi]}^{\text{D;G}}$ .

Let  $\pi' = (\pi_1, \mathbf{1}_{\kappa}, \text{fst}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa})$  and  $\pi'' = (\mathbf{1}_{\kappa}, \pi_2, \text{snd}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa})$ .

Note that, by Lemma 5.8,  $\text{trans}(\rho[\alpha \mapsto \pi'], \rho[\alpha \mapsto \pi'']) = \text{trans}(\rho, \rho')[\alpha \mapsto \pi]$ .

By IH applied to  $\Delta, \alpha :: \kappa \vdash \mathbf{t} :: *$ , we know  $\text{trans}(\mathcal{E}[\mathbf{t}]_{\rho[\alpha \mapsto \pi']}^{\text{D;G}}, \mathcal{E}[\mathbf{t}]_{\rho'[\alpha \mapsto \pi'']}^{\text{D;G}})_* \equiv_*^{\text{D;G}} \mathcal{E}[\mathbf{t}]_{\text{trans}(\rho, \rho')[\alpha \mapsto \pi]}^{\text{D;G}}$ .

Therefore, to show  $(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\mathbf{t}]_{\text{trans}(\rho, \rho')[\alpha::\kappa \mapsto \pi]}^{\text{D;G}}$ , it suffices to show

$$(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \text{trans}(\mathcal{E}[\mathbf{t}]_{\rho[\alpha \mapsto \pi']}^{\text{D;G}}, \mathcal{E}[\rho'[\alpha \mapsto \pi'']]_{\rho}^{\text{D;G}})_*.$$

By definition, it suffices to show

$$(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_2[\mathbf{1}_{\kappa}/\alpha]) \in \mathcal{E}[\mathbf{t}_2]_{\rho[\alpha \mapsto \pi']}^{\text{D;G}} \text{ and}$$

$$(\mathbf{m}'_2[\mathbf{1}_{\kappa}/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\mathbf{t}_2]_{\rho'[\alpha \mapsto \pi'']}^{\text{D;G}},$$

which follow from instantiating  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho}^{\text{D;G}}$  with  $\pi'$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{V}[\forall\alpha::\kappa. \mathbf{t}]_{\rho'}^{\text{D;G}}$  with  $\pi''$ .

$$\text{Case (FK-Fun)} \frac{\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2}{\Delta \vdash \lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2}$$

We must show

$$\text{trans}(\mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\text{D;G}}, \mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\text{D;G}})_{\kappa_1 \rightarrow \kappa_2} \equiv_{\kappa_1 \rightarrow \kappa_2}^{\text{D;G}} \mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}}.$$

By definition, it suffices to show

$$\begin{aligned} & \lambda_{\mathbf{R}}(\mathbf{t}_1, \mathbf{t}_3, \mathbf{R}).\text{trans}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho}^{\text{D;G}} \rho[\alpha :: \kappa_1 \mapsto (\mathbf{t}_1, \mathbf{1}_{\kappa_1}, \text{fst}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})], \\ & \quad \mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho}^{\text{D;G}} \rho[\alpha :: \kappa_1 \mapsto (\mathbf{1}_{\kappa_1}, \mathbf{t}_2, \text{snd}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})])_{\kappa_2} \\ & \equiv_{\kappa_1 \rightarrow \kappa_2}^{\text{D;G}} \mathcal{T}[\lambda\alpha::\kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}}. \end{aligned}$$

For arbitrary  $\pi \in \text{Rel}_{\kappa}^{\text{D;G}}$ , it suffices to show

$$\text{trans}(\mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho[\alpha::\kappa_1 \mapsto (\pi_1, \mathbf{1}_{\kappa_1}, \text{fst}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa_1})]}^{\text{D;G}}, \mathcal{T}[\mathbf{t} :: \kappa_2]_{\rho'[\alpha::\kappa_1 \mapsto (\mathbf{1}_{\kappa_1}, \pi_2, \text{snd}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa_1})]}^{\text{D;G}})_{\kappa_2} \equiv_{\kappa_1 \rightarrow \kappa_2}^{\text{D;G}} \mathcal{T}[\mathbf{t} :: \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}}$$

which follows by IH applied to  $\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$ , and Lemma 5.8.

$$\text{Case (FK-App)} \frac{\Delta \vdash \mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2 \quad \Delta \vdash \mathbf{t}_2 :: \kappa_1}{\Delta \vdash \mathbf{t}_1 \mathbf{t}_2 :: \kappa_2}$$

We must show  $\text{trans}(\mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\rho}^{\text{D;G}}, \mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\rho'}^{\text{D;G}})_{\kappa_2} \equiv_{\kappa_2}^{\text{D;G}} \mathcal{T}[\mathbf{t}_1 \mathbf{t}_2 :: \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}}$ .

Let  $\pi = (\text{trans}(\rho, \rho') \mathbf{1} \mathbf{t}_2, \text{trans}(\rho, \rho') \mathbf{2} \mathbf{t}_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}})$ ,  $\pi' = (\rho_1 \mathbf{t}_2, \rho_2 \mathbf{t}_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho}^{\text{D;G}})$ , and  $\pi'' = (\rho'_1 \mathbf{t}_2, \rho'_2 \mathbf{t}_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho'}^{\text{D;G}})$ .

By definition, it suffices to show  $\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\text{D;G}} \pi', \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\text{D;G}} \pi'')_{\kappa_2} \equiv_{\kappa_2}^{\text{D;G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}} \pi$

By the IH applied to  $\Delta \vdash \mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2$  we know

$$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\text{D;G}}, \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\text{D;G}})_{\kappa_1 \rightarrow \kappa_2} \equiv_{\kappa_1 \rightarrow \kappa_2}^{\text{D;G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}}$$

By definition of  $\equiv_{\kappa_1 \rightarrow \kappa_2}^{\text{D;G}}$ , we know

$$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\text{D;G}}, \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\text{D;G}})_{\kappa_1 \rightarrow \kappa_2} \pi \equiv_{\kappa_2}^{\text{D;G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\text{D;G}} \pi$$

By definition, we know

$$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\text{D;G}}, \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\text{D;G}})_{\kappa_1 \rightarrow \kappa_2} =$$

$\lambda_R(\mathbf{t}, \mathbf{t}', \mathbf{R}).\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}(\mathbf{t}, \mathbf{1}_{\kappa_2}, \text{fst}_R(\mathbf{R})), \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}(\mathbf{1}_{\kappa_2}, \mathbf{t}', \text{snd}_R(\mathbf{R})))_{\kappa_2}$

Therefore,

$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}(\pi_1, \mathbf{1}_{\kappa_2}, \text{fst}_R(\pi_R)), \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}(\mathbf{1}_{\kappa_2}, \pi_2, \text{snd}_R(\pi_R)))_{\kappa_2} \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}} \pi$

It suffices to show

$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}(\pi_1, \mathbf{1}_{\kappa_2}, \text{fst}_R(\pi_R)), \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}(\mathbf{1}_{\kappa_2}, \pi_2, \text{snd}_R(\pi_R)))_{\kappa_2} \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \pi', \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}} \pi'')_{\kappa_2}$ .

By IH applied to  $\Delta \vdash \mathbf{t}_2 :: \kappa_2$ , we know  $\text{trans}(\mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}})_{\kappa_2} \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\text{trans}(\rho, \rho')}^{\mathbf{D};\mathbf{G}}$ .

Therefore, by Lemma 5.9, we know  $\text{fst}_R(\pi_R) = \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}$ , and by Lemma 5.10 we know  $\text{snd}_R(\pi_R) = \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}$ .

Since  $\text{trans}$  ignores the positions  $\mathbf{1}_{\kappa_2}$ ,  $\pi'_2$ , and  $\pi''_1$ , it follows that

$\text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}(\pi_1, \mathbf{1}_{\kappa_2}, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}}), \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}(\mathbf{1}_{\kappa_2}, \pi_2, \mathcal{T}[\mathbf{t}_2 :: \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}}))_{\kappa_2} \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \text{trans}(\mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho}^{\mathbf{D};\mathbf{G}} \pi', \mathcal{T}[\mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2]_{\rho'}^{\mathbf{D};\mathbf{G}} \pi'')_{\kappa_2}$ .

□

### Lemma 5.12 (Lattice Interp. Self-Trans)

Let  $\rho = \llbracket \mathcal{L}_{\ell}^+ \rrbracket_{\zeta}^{\Sigma}$ .  $\text{trans}(\rho, \rho) = \rho$

### Lemma 5.13 (Target Relation Transitivity under Lattice Interp.)

Let  $\rho = \llbracket \mathcal{L}_{\ell}^+ \rrbracket_{\zeta}^{\Sigma}$ .

If  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$  and  $(\mathbf{m}_2, \mathbf{m}_3) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$  then  $(\mathbf{m}_1, \mathbf{m}_3) \in \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$ .

#### Proof

By Lemma 5.11, we know  $\text{trans}(\mathcal{E}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}, \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma})_* \equiv_*^{\mathbf{D};\mathbf{G}} \mathcal{E}[\mathbf{t}]_{\text{trans}(\rho, \rho)}^{\Sigma}$ .

By Lemma 5.12, we know  $\text{trans}(\rho, \rho) = \rho$ , therefore  $\text{trans}(\mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}, \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma})_* \equiv_*^{\Sigma} \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}$ .

The result follows by definition of  $\text{trans}(\mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma}, \mathcal{E}[\mathbf{t}]_{\rho}^{\Sigma})_*$ . □

## 5.3 Parametricity

### Theorem 5.14 (Fundamental Property)

If  $\Delta; \Gamma \vdash \mathbf{m} : \mathbf{t}$  then  $\Delta; \Gamma \vdash \mathbf{m} \approx \mathbf{m} : \mathbf{t}$

#### Proof

The structure of proof is similar to (but less complex than) Lemma 5.34 (Fundamental Property of Back-translation), and the work of Vytiniotis and Weirich [4].

We omit the statement of key lemmas as they are already well explained by Vytiniotis and Weirich [4]. □

### Lemma 5.15 (Preservation)

If  $\Delta; \Gamma \vdash \mathbf{m} : \mathbf{t}$  and  $\mathbf{m} \mapsto^* \mathbf{m}'$  then  $\Delta; \Gamma \vdash \mathbf{m}' : \mathbf{t}$

### Lemma 5.16 (Canonical Forms)

Let  $\Delta; \Gamma \vdash \mathbf{m} : \mathbf{t}$ .

- If  $\mathbf{t} = \alpha \in \Delta$ ,  $\mathbf{m} \mapsto^* \mathbf{m}'$  and  $\text{irred}(\mathbf{m}')$ .
- If  $\mathbf{t} = \mathbf{1}$ ,  $\mathbf{m} \mapsto^* \langle \rangle$ .

- If  $\mathbf{t} = \mathbf{t}_1 \times \mathbf{t}_2$ ,  $\mathbf{m} \mapsto^* \langle \mathbf{m}_1, \mathbf{m}_2 \rangle$ .
- If  $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ ,  $\mathbf{m} \mapsto^* \text{inj}_i \mathbf{m}'$ .
- If  $\mathbf{t} = \mathbf{t}_1 \rightarrow \mathbf{t}_2$ ,  $\mathbf{m} \mapsto^* \lambda x:\mathbf{t}_1. \mathbf{m}'$ .
- If  $\mathbf{t} = \forall \alpha::\kappa. \mathbf{t}'$ ,  $\mathbf{m} \mapsto^* \Lambda \alpha::*. \mathbf{m}'$ .

**Proof**

Follows by Theorem 5.14 (Fundamental Property) and the definition of  $\mathcal{V}[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}}$ . □

**Lemma 5.17 (Free Theorem: Parametric Condition)**

$\mathbf{D}, \alpha :: * \vdash \rho_i(\mathbf{t}_i) :: *, \mathbf{D} \vdash \rho_1(\mathbf{t}_g) :: *, \mathbf{D} \vdash \rho_2(\mathbf{t}_f) :: *,$

$\mathbf{D}; \mathbf{G} \vdash \mathbf{m} : \rho_i(\forall \alpha::*. (\mathbf{t}_1 \times (\mathbf{t}_2 \rightarrow \alpha)) \rightarrow \alpha),$

$\mathbf{D}; \mathbf{G} \vdash \mathbf{m}_f : \mathbf{t}_g \rightarrow \mathbf{t}_f, \mathbf{D}; \mathbf{G} \vdash \mathbf{m}_g : \mathbf{t}_2 \rightarrow \mathbf{t}_g,$

$(\mathbf{m}_a, \mathbf{m}'_a) \in \mathcal{E}[\mathbf{t}_1]_{\rho[\alpha \mapsto (\mathbf{t}_g, \mathbf{t}_f, \mathbf{R})]}^{\mathbf{D};\mathbf{G}},$  then

$(\mathbf{m}_f(\mathbf{m}[\mathbf{t}_g] \langle \mathbf{m}_a, \mathbf{m}_g \rangle), \mathbf{m}[\mathbf{t}_f] \langle \mathbf{m}'_a, \mathbf{m}_f \circ \mathbf{m}_g \rangle) \in \mathcal{E}[\mathbf{t}_f]_{\rho}^{\mathbf{D};\mathbf{G}}$

**Lemma 5.18 (Free Theorem:  $\eta_k^{\ell, s}$  shuffling)**

Let  $\rho = \llbracket \mathcal{L}_k^+ \rrbracket_{\zeta}^{\Sigma}$ .

If  $\Sigma \vdash \Lambda \beta::*. \mathbf{m}_1 : \mathbb{T}_\ell \mathbf{s}^{\dagger}$  then  $(\Lambda \beta::*. \mathbf{m}_1, \mathbf{m}_1[(\mathbb{T}_\ell \mathbf{s})^{\dagger} / \beta] \langle \hat{\text{pf}}[\ell \leq \mathbb{T}_\ell \mathbf{s}], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}^+]_{\rho}^{\Sigma}$

**Proof**

Must show:  $(\Lambda \beta::*. \mathbf{m}_1, \mathbf{m}[(\mathbb{T}_\ell \mathbf{s})^{\dagger} / \beta] \langle \hat{\text{pf}}[\ell \leq \mathbb{T}_\ell \mathbf{s}], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}^+]_{\rho}^{\Sigma}$

Have:  $\mathbf{m}_1[(\mathbb{T}_\ell \mathbf{s})^{\dagger} / \beta] \langle \hat{\text{pf}}[\ell \leq \mathbb{T}_\ell \mathbf{s}], \eta_k^{\ell, s} \rangle \mapsto^* \Lambda \beta::*. \mathbf{m}_2$  by Lemma 5.16 (Canonical Forms).

Must show:  $(\Lambda \beta::*. \mathbf{m}_1, \Lambda \beta::*. \mathbf{m}_2) \in \mathcal{V}[\mathbb{T}_\ell \mathbf{s}^+]_{\rho}^{\Sigma}$

Consider arbitrary  $\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_\beta$  such that  $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_\beta) \in \text{Rel}_{*}^{\Sigma}$ .

Let  $\rho' = \rho[\beta \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_\beta)]$ .

Must show:  $(\mathbf{m}_1[\mathbf{t}_1 / \beta], \mathbf{m}_2[\mathbf{t}_2 / \beta]) \in \mathcal{E}[\llbracket (\alpha_{\leq} \alpha_{\ell} \beta) \times (\mathbf{s}^{\dagger} \rightarrow \beta) \rrbracket \rightarrow \beta]_{\rho'}^{\Sigma}$

Have:  $\mathbf{m}_1[\mathbf{t}_1 / \beta] \mapsto^* \lambda x:((\hat{\alpha}_{\leq} \hat{\alpha}_{\ell} \mathbf{t}_1) \times (\mathbf{s}^{\dagger} \rightarrow \mathbf{t}_1)). \mathbf{m}_{11}$  by Lemma 5.16 (Canonical Forms).

Have:  $\mathbf{m}_2[\mathbf{t}_2 / \beta] \mapsto^* \lambda x:((\hat{\alpha}_{\leq} \hat{\alpha}_{\ell} \mathbf{t}_2) \times (\mathbf{s}^{\dagger} \rightarrow \mathbf{t}_2)). \mathbf{m}_{22}$  by Lemma 5.16 (Canonical Forms).

Must show:  $(\lambda x:((\hat{\alpha}_{\leq} \hat{\alpha}_{\ell} \mathbf{t}_1) \times (\mathbf{s}^{\dagger} \rightarrow \mathbf{t}_1)). \mathbf{m}_{11}, \lambda x:((\hat{\alpha}_{\leq} \hat{\alpha}_{\ell} \mathbf{t}_2) \times (\mathbf{s}^{\dagger} \rightarrow \mathbf{t}_2)). \mathbf{m}_{22}) \in \mathcal{V}[\llbracket (\alpha_{\leq} \alpha_{\ell} \beta) \times (\mathbf{s}^{\dagger} \rightarrow \beta) \rrbracket \rightarrow \beta]_{\rho'}^{\Sigma}$

(1) Consider arbitrary  $\mathbf{m}'_1, \mathbf{m}'_2$  such that  $(\mathbf{m}'_1, \mathbf{m}'_2) \in \mathcal{E}[\llbracket (\alpha_{\leq} \alpha_{\ell} \beta) \times (\mathbf{s}^{\dagger} \rightarrow \beta) \rrbracket]_{\rho'}^{\Sigma}$ .

Must show:  $(\mathbf{m}_{11}[\mathbf{m}'_1 / \mathbf{x}], \mathbf{m}_{22}[\mathbf{m}'_2 / \mathbf{x}]) \in \mathcal{E}[\llbracket \beta \rrbracket]_{\rho'}^{\Sigma}$

Assume that  $\ell \not\sqsubseteq \zeta$ :

Have:  $(\mathbf{m}_{11}[\mathbf{m}'_1 / \mathbf{x}], \mathbf{m}_{22}[\mathbf{m}'_2 / \mathbf{x}]) \in \text{Atom}[\llbracket \beta \rrbracket]_{\rho'}^{\Sigma}$  by assumption.

Have:  $\mathbf{m}_{11}[\mathbf{m}'_1 / \mathbf{x}] \mapsto^* \mathbf{m}_{1ir}$  and  $\text{irred}(\mathbf{m}_{1ir})$  by Lemma 5.16 (Canonical Forms).

Have:  $\mathbf{m}_{22}[\mathbf{m}'_2 / \mathbf{x}] \mapsto^* \mathbf{m}_{2ir}$  and  $\text{irred}(\mathbf{m}_{2ir})$  by Lemma 5.16 (Canonical Forms).

Must show:  $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathcal{V}[\llbracket \beta \rrbracket]_{\rho'}^{\Sigma}$ .

Must show:  $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathbf{R}_\beta$

Have:  $(\text{prj}_1 \mathbf{m}'_1, \text{prj}_1 \mathbf{m}'_2) \in \mathcal{E}[\llbracket (\alpha_{\leq} \alpha_{\ell} \beta) \rrbracket]_{\rho'}^{\Sigma}$  from 1.

Have:  $\ell \not\sqsubseteq \zeta \implies \mathbf{R}_\beta = \text{Atom}[\mathbf{t}_1, \mathbf{t}_2]_{\rho}^{\Sigma}$  by definition of  $\rho$  on  $\alpha_{\leq}$  and previous fact.

Have:  $\mathbf{R}_\beta = \text{Atom}[\mathbf{t}_1, \mathbf{t}_2]^\Sigma$  by assumption  $\ell \not\sqsubseteq \zeta$  and previous fact.

Have:  $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \text{Atom}[\beta]_{\rho'}^\Sigma$  by Lemma 5.15 (Preservation).

Have:  $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \text{Atom}[\mathbf{t}_1, \mathbf{t}_2]^\Sigma$  by definition of  $\rho'$ .

Therefore:  $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathbf{R}_\beta$ .

Assume that  $\ell \sqsubseteq \zeta$ :

(2) Have:  $(\Lambda\beta::*\mathbf{m}_1, \Lambda\beta::*\mathbf{m}_1) \in \mathcal{V}[\mathbb{T}_\ell s^+]_{\rho}^\Sigma$  by Theorem 5.14. (Fundamental Property)

$$\text{Let } \mathbf{R} = \left\{ \begin{array}{l} (\mathbf{m}_1, \mathbf{m}_2) \in \text{Atom}[\mathbf{t}_1, \mathbb{T}_\ell s^\dagger]^\Sigma \mid \\ \ell \sqsubseteq \zeta \implies \exists \mathbf{m}, \mathbf{m}_2 \mapsto^* \Lambda\beta::*\mathbf{m} \wedge (\mathbf{m}_1, \mathbf{m}[\mathbf{t}_2] \mathbf{m}_2) \in \mathbf{R}_\beta \end{array} \right\}.$$

Observe that:  $\rho$  on  $\alpha_{\leq}$  requires that if  $\ell \not\sqsubseteq \zeta$ , then  $\mathbf{R} = \text{Atom}[\mathbf{t}_1, \mathbb{T}_\ell s^\dagger]^\Sigma$ , hence the  $\ell \sqsubseteq \zeta$  condition.

Let  $\rho'' = \rho[\beta \mapsto (\mathbf{t}_1, \mathbb{T}_\ell s^\dagger, \mathbf{R})]$ .

Instantiating 2 with  $(\mathbf{t}_1, \mathbb{T}_\ell s^+, \mathbf{R})$ :

$$(\mathbf{m}_1[\mathbf{t}_1/\beta], \mathbf{m}_1[\mathbb{T}_\ell s^\dagger/\beta]) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta) \rightarrow \beta]_{\rho''}^\Sigma,$$

Now we will show:  $(\mathbf{m}'_1, \langle \hat{\text{pf}}[\ell \leq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta)]_{\rho''}^\Sigma,$

Recall:  $(\mathbf{m}'_1, \mathbf{m}'_2) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta)]_{\rho'}^\Sigma.$

Therefore:  $(\text{prj}_1 \mathbf{m}'_1, \text{prj}_1 \mathbf{m}'_2) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta)]_{\rho'}^\Sigma.$

Observe that:  $(\text{prj}_1 \mathbf{m}'_1, \hat{\text{pf}}[\ell \leq \mathbb{T}_\ell s]) \in \text{Atom}[\mathbf{t}_1, \mathbb{T}_\ell s^\dagger]^\Sigma.$

Have:  $(\text{prj}_1 \mathbf{m}'_1, \text{pf}[\ell \leq \mathbb{T}_\ell s]) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta)]_{\rho''}^\Sigma$  by previous fact, definition of  $\mathbf{R}$ , and definition of  $\rho$  on  $\alpha_{\leq}$ .

Must show:  $(\text{prj}_2 \mathbf{m}'_1, \eta_k^{\ell, s}) \in \mathcal{E}[s^+ \rightarrow \beta]_{\rho''}^\Sigma,$

Assume that  $(\mathbf{m}_{s1}, \mathbf{m}_{s2}) \in \mathcal{E}[s^+]_{\rho''}^\Sigma:$

Must show:  $((\text{prj}_2 \mathbf{m}'_1) \mathbf{m}_{s1}, \eta_k^{\ell, s} \mathbf{m}_{s2}) \in \mathbf{R}$  Have:  $\ell \sqsubseteq \zeta$  by assumption.

Must show:  $\exists \mathbf{m}, \eta_k^{\ell, s} \mathbf{m}_{s2} \mapsto^* \Lambda\beta::*\mathbf{m} \wedge ((\text{prj}_2 \mathbf{m}'_1) \mathbf{m}_{s1}, \mathbf{m}[\mathbf{t}_2] \mathbf{m}'_2) \in \mathbf{R}_\beta$

Observe that:  $\eta_k^{\ell, s} \mathbf{m}_{s2} \mapsto^* \Lambda\beta::*\lambda\mathbf{x}:((\hat{\alpha}_{\leq} \hat{\alpha}_\ell \beta) \times s^+ \rightarrow \beta).((\text{prj}_2 \mathbf{x}) \mathbf{m}_{s2}).$

Therefore:  $\exists \mathbf{m} = (\lambda\mathbf{x}:((\hat{\alpha}_{\leq} \hat{\alpha}_\ell \beta) \times s^\dagger \rightarrow \beta).((\text{prj}_2 \mathbf{x}) \mathbf{m}_{s2})).$

$\eta_k^{\ell, s} \mathbf{m}_{s2} \mapsto^* \Lambda\beta::*\mathbf{m}.$

Must show:  $((\text{prj}_2 \mathbf{m}'_1) \mathbf{m}_{s1}, \mathbf{m}[\mathbf{t}_2] \mathbf{m}'_2) \in \mathbf{R}_\beta$

Observe that:  $(\lambda\mathbf{x}:((\hat{\alpha}_{\leq} \hat{\alpha}_\ell \beta) \times s^\dagger \rightarrow \beta).((\text{prj}_2 \mathbf{x}) \mathbf{m}_{s2}))[\mathbf{t}_2] \mathbf{m}'_2 \mapsto^* (\text{prj}_2 \mathbf{m}'_2) \mathbf{m}_{s2}.$

Must show:  $((\text{prj}_2 \mathbf{m}'_1) \mathbf{m}_{s1}, (\text{prj}_2 \mathbf{m}'_2) \mathbf{m}_{s2}) \in \mathbf{R}_\beta$

Recall:  $(\mathbf{m}'_1, \mathbf{m}'_2) \in \mathcal{E}[(\alpha_{\leq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta)]_{\rho'}^\Sigma$

Therefore:  $(\text{prj}_2 \mathbf{m}'_1, \text{prj}_2 \mathbf{m}'_2) \in \mathcal{E}[s^+ \rightarrow \beta]_{\rho'}^\Sigma.$

Have:  $((\text{prj}_2 \mathbf{m}'_1) \mathbf{m}_{s1}, \text{prj}_2 \mathbf{m}'_2 \mathbf{m}_{s2}) \in \mathbf{R}_\beta$  by assumption that  $(\mathbf{m}_{s1}, \mathbf{m}_{s2}) \in \mathcal{E}[s^+]_{\rho''}^\Sigma.$

Therefore:  $(\mathbf{m}_1[t_1/\beta] \mathbf{m}'_1, \mathbf{m}_1[\mathbb{T}_\ell s^\dagger/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\beta]_{\rho''}^\Sigma$ .

Recall:  $\mathbf{m}_1[\mathbb{T}_\ell s^\dagger/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle \mapsto^* \Lambda\beta::*\mathbf{m}_2$  and  $\mathbf{m}_1[t_1/\beta] \mathbf{m}'_1 \mapsto^* \mathbf{m}_{11}[\mathbf{m}'_1/x]$ .

Have:  $(\mathbf{m}_{11}[\mathbf{m}'_1/x], \mathbf{m}_2[t_2] \mathbf{m}'_2) \in \mathbf{R}_\beta$  by definition of  $\mathbf{R}$ .

Recall:  $\mathbf{m}_2[t_2] \mathbf{m}'_2 \mapsto^* \mathbf{m}_{22}[\mathbf{m}'_2/x]$ .

Therefore:  $(\mathbf{m}_{11}[\mathbf{m}'_1/x], \mathbf{m}_{22}[\mathbf{m}'_2/x]) \in \mathcal{E}[\beta]_{\rho'}^\Sigma$ .

□

## 5.4 Translation

### Lemma 5.19

If  $\ell \preceq s$  then  $\mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+ \vdash \text{pf}[\ell \preceq s] : (\alpha_\preceq \ \alpha_\ell \ s^+)$  and  $\mathbf{D}_\ell; \mathbf{G}_\ell, \mathbf{G}_\preceq \vdash \text{pf}[\ell \preceq s] : (\hat{\alpha}_\preceq \ \hat{\alpha}_\ell \ s^\dagger)$

### Lemma 5.20 (Translation is Complete)

If  $\Gamma \vdash e : s$  then  $\exists \mathbf{m}. \Gamma \vdash e : s \rightsquigarrow \mathbf{m}$

### Theorem 5.21 (Translation Preserves Well-Typedness)

If  $\Gamma \vdash e : s$  and  $\Gamma \vdash e : s \rightsquigarrow \mathbf{m}$  then  $\mathcal{L}_\ell^+; \mathcal{L}_\square^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : s^+$

### Proof

Proof by induction on the translation derivation,  $\Gamma \vdash e : s \rightsquigarrow \mathbf{m}$ .

□

### Lemma 5.22 (Cross Lang. Relation Respects Target Relation)

Let  $\rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma$  and  $\delta = \rho_i$  (since  $\rho_1 = \rho_2$ ).

1. If  $(e, \mathbf{m}) \in \mathcal{E}_\zeta^+[\mathbf{s}]_\delta$  and  $(\mathbf{m}, \mathbf{m}') \in \mathcal{E}[\mathbf{s}^+]_\rho^\Sigma$  then  $(e, \mathbf{m}') \in \mathcal{E}_\zeta^+[\mathbf{s}]_\delta$
2. If  $(v, \mathbf{u}) \in \mathcal{V}_\zeta^+[\mathbf{s}]_\delta$  and  $(\mathbf{u}, \mathbf{u}') \in \mathcal{V}[\mathbf{s}^+]_\rho^\Sigma$  then  $(v, \mathbf{u}') \in \mathcal{V}_\zeta^+[\mathbf{s}]_\delta$

### Proof

We prove 1 and 2 simultaneously. Proof by induction on the structure of the type  $s$ .

We give the cases for  $s_1 \rightarrow s_2$  and  $\mathbb{T}_\ell s_1$ .

#### Case 1

Must show:  $(e, \mathbf{m}') \in \mathcal{E}_\zeta^+[\mathbf{s}]_\delta$ .

By definition, it suffices to show  $e \mapsto^* v$ ,  $\mathbf{m}' \mapsto^* \mathbf{u}$  and  $(v, \mathbf{u}) \in \mathcal{V}_\zeta^+[\mathbf{s}]_\delta$ .

Have:  $e \mapsto^* v$  from  $(e, \mathbf{m}) \in \mathcal{E}_\zeta^+[\mathbf{s}]_\delta$ .

Have:  $\mathbf{m}' \mapsto^* \mathbf{u}'$  from  $(\mathbf{m}, \mathbf{m}') \in \mathcal{E}[\mathbf{s}^+]_\rho^\Sigma$ .

Have:  $(v, \mathbf{u}') \in \mathcal{V}_\zeta^+[\mathbf{s}]_\delta$  by 2.

#### Case 2, $s_1 \rightarrow s_2$

Must show:  $(\lambda x : s_1. e_2, \lambda x : s_1^+. \mathbf{m}'_2) \in \mathcal{V}_\zeta^+[\mathbf{s}_1 \rightarrow \mathbf{s}_2]_\delta$

Consider arbitrary  $e'_1, \mathbf{m}'_1$  such that  $(e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$ .

Must show:  $(e_2[e'_1/x], \mathbf{m}'_2[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+[\mathbf{s}_2]_\delta$

Instantiating  $(\lambda x : s_1. e_2, \lambda x : s_1^+. \mathbf{m}_2) \in \mathcal{V}_\zeta^+[\mathbf{s}_1 \rightarrow \mathbf{s}_2]_\delta$  with  $(e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$ :

Therefore:  $(e_2[e'_1/x], \mathbf{m}_2[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+[\mathbf{s}_2]_\delta$ .

Instantiating  $(\lambda x : s_1. e_2, \lambda x : s_1^+. \mathbf{m}_2) \in \mathcal{V}_\zeta^+[\mathbf{s}_1 \rightarrow \mathbf{s}_2]_\delta$  with  $(e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$ :

Therefore:  $(e_2[e'_1/x], \mathbf{m}_2[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+[\mathbf{s}_2]_\delta$ .



Have:  $(\mathbf{m}'_1, \mathbf{m}'_1) \in \mathcal{E}[\mathbf{s}_2^+]_\rho^\Sigma$  by Theorem 5.14 (Fundamental Property).  
 Instantiating  $(\lambda \mathbf{x}: \mathbf{s}_1^+ . \mathbf{m}_2, \lambda \mathbf{x}: \mathbf{s}_1^+ . \mathbf{m}'_2) \in \mathcal{V}[\mathbf{s}_1^+ \rightarrow \mathbf{s}_2^+]_\rho^\Sigma$  with  $(\mathbf{m}'_1, \mathbf{m}'_1) \in \mathcal{E}[\mathbf{s}_1^+]_\rho^\Sigma$ :  
 Therefore:  $(\mathbf{m}_2[\mathbf{m}'_1/\mathbf{x}], \mathbf{m}'_2[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}[\mathbf{s}_2^+]_\rho^\Sigma$ .  
 Observe that:  $(\mathbf{e}_2[\mathbf{e}'_1/\mathbf{x}], \mathbf{m}_2[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\mathbf{s}_2]_\delta$ , and  $(\mathbf{m}_2[\mathbf{m}'_1/\mathbf{x}], \mathbf{m}'_2[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}[\mathbf{s}_2^+]_\rho^\Sigma$ .  
 Have:  $(\mathbf{e}_2[\mathbf{e}'_1/\mathbf{x}], \mathbf{m}'_2[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\mathbf{s}_2]_\delta$  by induction via part 1.

**Case 2,  $\mathbb{T}_\ell \mathbf{s}_1$**

Have:  $(\eta_\ell \mathbf{e}_1, \Lambda \beta::* . \mathbf{m}_{11}) \in \mathcal{V}_\zeta^+[\mathbb{T}_\ell \mathbf{s}_1]_\delta$ .  
 Have:  $(\Lambda \beta::* . \mathbf{m}_{11}, \Lambda \beta::* . \mathbf{m}'_{11}) \in \mathcal{V}[(\mathbb{T}_\ell \mathbf{s}_1)^+]_\rho^\Sigma$ .  
 Must show:  $(\eta_\ell \mathbf{e}_1, \Lambda \beta::* . \mathbf{m}'_{11}) \in \mathcal{V}_\zeta^+[\mathbb{T}_\ell \mathbf{s}_1]_\delta$   
 Must show:  $\exists \mathbf{m}'_1. (\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}'_1) \in \mathcal{E}[(\mathbb{T}_\ell \mathbf{s}_1)^+]_\rho^\Sigma$  and  
 $(\mathbf{e}_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$   
 Have:  $\exists \mathbf{m}_1. (\mathbf{m}_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}_1) \in \mathcal{E}[(\mathbb{T}_\ell \mathbf{s}_1)^+]_\rho^\Sigma$  and  
 $(\mathbf{e}_1, \mathbf{m}_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$  from  $(\eta_\ell \mathbf{e}_1, \Lambda \beta::* . \mathbf{m}_{11}) \in \mathcal{V}_\zeta^+[\mathbb{T}_\ell \mathbf{s}_1]_\delta$ .  
 Now we will show:  $\exists \mathbf{m}'_1. (\mathbf{m}_1, \mathbf{m}'_1) \in \mathcal{E}[\mathbf{s}_1]_\rho^\Sigma$  and  $(\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}'_1) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}_1^+]_\rho^\Sigma$

Take  $\mathbf{m}'_1$  to be  $\mathbf{m}_1$ .

Have:  $(\mathbf{m}_1, \mathbf{m}_1) \in \mathcal{E}[\mathbf{s}_1]_\rho^\Sigma$  by Theorem 5.14 (Parametricity).

Must show:  $(\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}_1) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}_1^+]_\rho^\Sigma$

Let  $\rho' = (\mathbb{T}_\ell \mathbf{s}^+, \mathbb{T}_\ell \mathbf{s}^+, \mathcal{E}[\mathbb{T}_\ell \mathbf{s}^+]_\rho^\Sigma)$ .

Instantiating  $(\Lambda \beta::* . \mathbf{m}_{11}, \Lambda \beta::* . \mathbf{m}'_{11}) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}^+]_\rho^\Sigma$  with  $(\mathbb{T}_\ell \mathbf{s}^+, \mathbb{T}_\ell \mathbf{s}^+, \mathcal{E}[\mathbb{T}_\ell \mathbf{s}^+]_\rho^\Sigma)$ :

$(\mathbf{m}_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta], \mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta]) \in \mathcal{E}[\langle (\alpha \preceq \alpha_\ell \beta) \times \mathbf{s}^+ \rightarrow \beta \rangle \rightarrow \beta]_{\rho'}^\Sigma$

Therefore:  $(\mathbf{m}_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\beta]_{\rho'}^\Sigma$ .

Therefore:  $(\mathbf{m}_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\mathbb{T}_\ell \ell^+]_\rho^\Sigma$ .

Have:  $(\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \mathbf{m}_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\mathbb{T}_\ell \ell^+]_\rho^\Sigma$  by Lemma 5.7 (Symmetry).

Have:  $(\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}_1) \in \mathcal{E}[\mathbb{T}_\ell \ell^+]_\rho^\Sigma$  by Lemma 5.13 (Transitivity).

Have:  $\exists \mathbf{m}'_1. (\mathbf{m}_1, \mathbf{m}'_1) \in \mathcal{E}[\mathbf{s}_1]_\rho^\Sigma$  and  $(\mathbf{m}'_{11}[(\mathbb{T}_\ell \mathbf{s}_1)^+/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell \mathbf{s}_1], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}'_1) \in \mathcal{E}[\mathbb{T}_\ell \mathbf{s}_1^+]_\rho^\Sigma$ .

Have:  $(\mathbf{e}_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbf{s}_1]_\delta$  from the induction hypothesis (1) applied to  $\mathbf{s}_1$ .

□

### Theorem 5.23 (Translation Preserves Semantics)

If  $\Gamma \vdash \mathbf{e} : \mathbf{s}$  and  $\Gamma \vdash \mathbf{e} : \mathbf{s} \rightsquigarrow \mathbf{m}$  then  $\Gamma \vdash \mathbf{e} \simeq \mathbf{m} : \mathbf{s}$

#### Proof

By induction on the typing derivation,  $\Gamma \vdash \mathbf{e} : \mathbf{s}$ . There are two interesting cases. Other cases follow easily by induction.

By the definition of the cross language logical relation, all cases require us to prove  $\Gamma \vdash \mathbf{e} : \mathbf{s}$  and  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathbf{s}^+$ . We omit these below, since  $\Gamma \vdash \mathbf{e} : \mathbf{s}$  follows by assumption and  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathbf{s}^+$  follows by Theorem 5.21, Translation Preserves Well-Typedness.

**Case DT-Prot**

Have:  $\Gamma \vdash \eta_\ell e : \mathbb{T}_\ell s$ .

Have:  $\Gamma \vdash \eta_\ell e : \Lambda\beta::*\lambda x:((\alpha_{\preceq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta)).((\text{prj}_2 x) m) \rightsquigarrow \mathbb{T}_\ell s$ .

Must show:  $\Gamma \vdash \eta_\ell e \simeq \Lambda\beta::*\lambda x:((\alpha_{\preceq} \alpha_\ell \beta) \times (s^+ \rightarrow \beta)).((\text{prj}_2 x) m) : \mathbb{T}_\ell s$

Consider arbitrary  $\delta, \gamma, \Upsilon, \Upsilon_\square, \Upsilon_{\preceq}$  such that  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$ ,  $\Upsilon_\square = \llbracket \mathcal{L}_\square^+ \rrbracket$ ,  $\Upsilon_{\preceq} = \llbracket \preceq^+ \rrbracket$ , and  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+(\llbracket \Gamma \rrbracket)_\delta$ .

Let  $e_1 = \gamma(e)$ .

Let  $m_2 = \delta(\Upsilon_\square(\Upsilon_{\preceq}(\Upsilon(m))))$ .

Must show:  $(\eta_\ell e_1, \Lambda\beta::*\lambda x:((\hat{\alpha}_{\preceq} \hat{\alpha}_\ell \beta) \times s^{\hat{\dagger}} \rightarrow \beta)).((\text{prj}_2 x) m_2) \in \mathcal{E}_\zeta^+(\llbracket \mathbb{T}_\ell s \rrbracket)_\delta$

Must show:  $(\eta_\ell e_1, \Lambda\beta::*\lambda x:((\hat{\alpha}_{\preceq} \hat{\alpha}_\ell \beta) \times s^{\hat{\dagger}} \rightarrow \beta)).((\text{prj}_2 x) m_2) \in \mathcal{V}_\zeta^+(\llbracket \mathbb{T}_\ell s \rrbracket)_\delta$

Let  $t_{pf} = (\hat{\alpha}_{\preceq} \hat{\alpha}_\ell (\mathbb{T}_\ell s)^{\hat{\dagger}}) \times (s^{\hat{\dagger}} \rightarrow (\mathbb{T}_\ell s)^{\hat{\dagger}})$ .

Let  $\rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma$ .

Must show:  $\exists m'_2. (\lambda x:t_{pf}.((\text{prj}_2 x) m_2) \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} m'_2) \in \mathcal{E}[\llbracket \mathbb{T}_\ell s^+ \rrbracket_\rho^\Sigma]$  and  $(e_1, m'_2) \in \mathcal{V}_\zeta^+(\llbracket s \rrbracket)_\delta$

Take  $m'_2$  to be  $m_2$ .

Observe that:

$\lambda x:t_{pf}.((\text{prj}_2 x) m_2) \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle \mapsto^* \text{prj}_2 \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle m_2 \mapsto^* \eta_k^{\ell, s} m_2$ .

Therefore:  $(\lambda x:t_{pf}.((\text{prj}_2 x) m_2) \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} m_2) \in \mathcal{E}[\llbracket \mathbb{T}_\ell s^+ \rrbracket_\rho^\Sigma]$ .

Must show:  $(e_1, m_2) \in \mathcal{E}_\zeta^+(\llbracket s \rrbracket)_\delta$

Have:  $\Gamma \vdash e \simeq m : s$  by applying the induction hypothesis to  $s$ .

Instantiating  $\Gamma \vdash e \simeq m : s$  with  $\delta, \gamma, \Upsilon, \Upsilon_\square$ , and  $\Upsilon_{\preceq} : (\gamma(e), \delta\Upsilon_\square\Upsilon_{\preceq}\Upsilon(m)) \in \mathcal{E}_\zeta^+(\llbracket s \rrbracket)_\delta$ .

Recall the definition of  $e_1$  and  $m_2$ :  $(e_1, m_2) \in \mathcal{E}_\zeta^+(\llbracket s \rrbracket)_\delta$ .

**Case DT-Bind**

Have:  $\Gamma \vdash \text{bind } x = e \text{ in } e' : s'$ .

Have:  $\Gamma \vdash \text{bind } x = e \text{ in } e' : s' \rightsquigarrow m[(s')^{\hat{\dagger}}] \langle \hat{\text{pf}}[\ell \preceq s'], (\lambda x:s^{\hat{\dagger}}.m') \rangle$ .

Must show:  $\Gamma \vdash \text{bind } x = e \text{ in } e' \simeq m[(s')^{\hat{\dagger}}] \langle \hat{\text{pf}}[\ell \preceq s'], (\lambda x:s^{\hat{\dagger}}.m') \rangle : s'$

Consider arbitrary  $\delta, \gamma, \Upsilon, \Upsilon_\square$ , and  $\Upsilon_{\preceq}$  such that  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$ ,  $\Upsilon_\square = \llbracket \mathcal{L}_\square^+ \rrbracket$ ,  $\Upsilon_{\preceq} = \llbracket \preceq^+ \rrbracket$ , and  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+(\llbracket \Gamma \rrbracket)_\delta$ .

Let  $e_1 = \gamma(e)$ .

Let  $e_2 = \gamma(e')$ .

Let  $m_1 = \delta(\Upsilon_\square(\Upsilon_{\preceq}(\Upsilon(m))))$ .

Let  $m_2 = \delta(\Upsilon_\square(\Upsilon_{\preceq}(\Upsilon(m'))))$ .

Must show:  $(\text{bind } x = e_1 \text{ in } e_2, m_1[(s')^{\hat{\dagger}}] \langle \hat{\text{pf}}[\ell \preceq s'], (\lambda x:s^{\hat{\dagger}}.m_2) \rangle) \in \mathcal{E}_\zeta^+(\llbracket s_2 \rrbracket)_\delta$

Let  $m_f = \lambda y:(\mathbb{T}_\ell s)^{\hat{\dagger}}.y[(s')^{\hat{\dagger}}] \langle \hat{\text{pf}}[\ell \preceq s'], \lambda x:s^{\hat{\dagger}}.m_2 \rangle : \mathbb{T}_\ell s^{\hat{\dagger}} \rightarrow (s')^{\hat{\dagger}}$ .

Let  $m_g = \lambda y:s^{\hat{\dagger}}.\Lambda\beta::*\lambda x:((\hat{\alpha}_{\preceq} \hat{\alpha}_\ell \beta) \times s^{\hat{\dagger}} \rightarrow \beta).((\text{prj}_2 x) y) : s^{\hat{\dagger}} \rightarrow \mathbb{T}_\ell s^{\hat{\dagger}}$ .

Observe that:  $m_g = \eta_k^{\ell, s}$ , and  $(m_f \circ m_g, \lambda x:s^{\hat{\dagger}}.m_2) \in \mathcal{E}[\llbracket s^{\hat{\dagger}} \rightarrow (s')^{\hat{\dagger}} \rrbracket_\rho^\Sigma]$ .

Have:  $\Gamma \vdash e \simeq m : \mathbb{T}_\ell s$  from induction hypothesis applied to  $\Gamma \vdash e : \mathbb{T}_\ell s$ .

Instantiating  $\Gamma \vdash e \simeq m : \mathbb{T}_\ell s$  with  $\delta, \gamma, \Upsilon, \Upsilon_\square$ , and  $\Upsilon_{\preceq}$ :

Have:  $(e_1, m_1) \in \mathcal{E}_\zeta^+(\llbracket \mathbb{T}_\ell s \rrbracket)_\delta$ .

Therefore:  $e_1 \mapsto^* \eta_\ell e'_1$ ,  $m_1 \mapsto^* \Lambda\beta::*.m_{11}$ ,  $(\eta_\ell e'_1, \Lambda\beta::*.m_{11}) \in \mathcal{V}_\zeta^+(\llbracket \mathbb{T}_\ell s \rrbracket)_\delta$ .

Let  $\rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma$ .

Therefore:  $\exists m'_1$ . such that  $(m_{11}[(\mathbb{T}_\ell s)^{\hat{\dagger}}/\beta] \langle \hat{\text{pf}}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} m'_1) \in \mathcal{E}[\llbracket (\mathbb{T}_\ell s)^+ \rrbracket_\rho^\Sigma]$  and  $(e'_1, m'_1) \in \mathcal{E}_\zeta^+(\llbracket s \rrbracket)_\delta$ .

Have:  $(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_g \rangle, \mathbf{m}_g \mathbf{m}'_1) \in \mathcal{E}[(\mathbb{T}_\ell s)^\dagger]_\rho^\Sigma$  by definition of  $\mathbf{m}_g$  and reduction.

Therefore:  $(\mathbf{m}_f(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_g \rangle), \mathbf{m}_f(\mathbf{m}_g \mathbf{m}'_1))$ .

Have:  $(\mathbf{m}_f(\mathbf{m}_g \mathbf{m}'_1), \mathbf{m}_f(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_g \rangle))$  by Lemma 5.7 (Symmetry).

Now we will show:  $(\text{bind } x = e_1 \text{ in } e_2, \mathbf{m}_f(\mathbf{m}_g \mathbf{m}'_1)) \in \mathcal{E}_\zeta^+[s']_\delta$

Have:  $\text{bind } x = e_1 \text{ in } e_2 \mapsto^* \text{bind } x = \eta_\ell e'_1 \text{ in } e_2 \mapsto^1 e_2[e'_1/x]$ .

Have:  $\mathbf{m}_f(\mathbf{m}_g \mathbf{m}'_1) \mapsto^* \lambda x:s^\dagger. \mathbf{m}_2 \mathbf{m}'_1 \mapsto^1 \mathbf{m}_2[\mathbf{m}'_1/x]$ .

Must show:  $(e_2[e'_1/x], \mathbf{m}_2[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+[s']_\delta$

Have:  $\Gamma, x:s \vdash e' \simeq \mathbf{m}' : s'$  from induction hypothesis applied to  $\Gamma, x:s \vdash e' : s'$ .

Instantiating  $\Gamma, x:s \vdash e' \simeq \mathbf{m}' : s'$  with  $\delta, \gamma[x \mapsto e'_1], \gamma[x \mapsto \mathbf{m}'_1], \gamma_\square$ , and  $\gamma_\preceq$ :

Recall definition of  $e_1$  and  $\mathbf{m}_1$ .

Have:  $(e_1[e'_1/x], \mathbf{m}_1[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+[s']_\delta$  by Compositionality.

Have:  $(\text{bind } x = e_1 \text{ in } e_2, \mathbf{m}_f(\mathbf{m}_g \mathbf{m}'_1)) \in \mathcal{E}_\zeta^+[s']_\delta$ .

Have:  $(\text{bind } x = e_1 \text{ in } e_2, \mathbf{m}_f(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_g \rangle)) \in \mathcal{E}_\zeta^+[s']_\delta$  by Lemma 5.22 (Cross Lang. Relation Respects Target Relation).

Have:  $(\mathbf{m}_f(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_g \rangle), (\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq \mathbb{T}_\ell s], \mathbf{m}_f \circ \mathbf{m}_g \rangle)) \in \mathcal{E}[(s')^\dagger]_\rho^\Sigma$  by Lemma 5.17 (Parametric Condition).

Have:  $(\text{bind } x = e_1 \text{ in } e_2, (\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger] \langle \hat{\mathbf{p}}\mathbf{f}[\ell \preceq s'], \mathbf{m}_f \circ \mathbf{m}_g \rangle)) \in \mathcal{E}_\zeta^+[s']_\delta$  by Lemma 5.22 (Cross Lang. Relation Respects Target Relation).

□

## 5.5 Back-translation

### Lemma 5.24

If  $\Sigma \vdash \mathbf{m} : (\hat{\alpha}_{\preceq} \hat{\alpha}_{\ell} s^{\dagger})$  then  $\ell \preceq s$ .

### Lemma 5.25 (Monotonicity)

If  $(\mathbf{W}, \mathbf{m}) \in \mathbf{R}$ ,  $(\mathbf{t}, \mathbf{R}) \in \text{Rel}^{\uparrow \Sigma}_{*}$ , and  $\mathbf{W}' \supseteq \mathbf{W}$ , then  $(\mathbf{W}', \mathbf{m}) \in \mathbf{R}$ .

#### Proof

Follows by definition of  $\text{Rel}^{\uparrow \Sigma}_{*}$ . □

### Lemma 5.26 (Monotonicity of $\mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ )

If  $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ , and  $\mathbf{W}' \supseteq \mathbf{W}$  then  $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$

#### Proof

**Case**  $\exists s.t = s^{\dagger}$ .

For arbitrary  $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![s^{\dagger}]\!]_{\emptyset}^{\Sigma}$ ,  $\mathbf{W}' \supseteq \mathbf{W}$ , we must show that  $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![s^{\dagger}]\!]_{\emptyset}^{\Sigma}$  and  $\exists e.\Sigma; \mathbf{W}'_k; \mathbf{W}'_{\Gamma} \vdash \mathbf{m} : s^{\dagger} \uparrow e$ .

By definition of  $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![s^{\dagger}]\!]_{\emptyset}^{\Sigma}$ , we know  $\exists e.\Sigma; \mathbf{W}_k; \mathbf{W}_{\Gamma} \vdash \mathbf{m} : s^{\dagger} \uparrow e$ .

The extra variables of  $\mathbf{W}'$  make no difference in back-translation since they are not used in  $\mathbf{m}$ . Since  $\mathbf{m}$  is back-translatable under  $\mathbf{W}$ , it is also back-translatable under  $\mathbf{W}'$ .

That is, the same back-translation derivation gives us  $\Sigma; \mathbf{W}'_k; \mathbf{W}'_{\Gamma} \vdash \mathbf{m} : s^{\dagger} \uparrow e$ .

**Case**  $\nexists s.t = s^{\dagger}$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ , we must show  $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}'$  and  $(\mathbf{W}_1, \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ , it suffices to show that  $(\mathbf{W}_1, \mathbf{E}^{\#}[\mathbf{m}]) \in \mathcal{O}^{\uparrow}[\![s^{\dagger}]\!]_{\delta}^{\Sigma}$ , which follows from instantiating  $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$  with  $\mathbf{W}_1 \supseteq \mathbf{W}$  and  $(\mathbf{W}_1, \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ . □

### Lemma 5.27 (Monotonicity of $\mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ )

If  $(\mathbf{W}, \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ , and  $\mathbf{W}' \supseteq \mathbf{W}$ , then  $(\mathbf{W}', \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ .

#### Proof

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ , we must show  $(\mathbf{W}', \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}'$ ,  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{V}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^{\#}[\mathbf{m}]) \in \mathcal{O}^{\uparrow}[\![s^{\dagger}]\!]_{\delta}^{\Sigma}$ .

Clearly  $\mathbf{W}_1 \supseteq \mathbf{W}$ , so we instantiate the hypothesis  $(\mathbf{W}, \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$  with  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{V}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ .

Thus  $(\mathbf{W}_1, \mathbf{E}^{\#}[\mathbf{m}]) \in \mathcal{O}^{\uparrow}[\![s^{\dagger}]\!]_{\delta}^{\Sigma}$ . □

### Lemma 5.28 (Monotonicity of $\mathcal{G}^{\uparrow}[\![\Gamma]\!]_{\delta}^{\Sigma}$ )

If  $(\mathbf{W}, \gamma) \in \mathcal{G}^{\uparrow}[\![\Gamma]\!]_{\delta}^{\Sigma}$ , and  $\mathbf{W}' \supseteq \mathbf{W}$  then  $(\mathbf{W}', \gamma) \in \mathcal{G}^{\uparrow}[\![\Gamma]\!]_{\delta}^{\Sigma}$

#### Proof

Follows simply from Lemma 5.26 (Monotonicity of  $\mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]_{\delta}^{\Sigma}$ ). □

### Lemma 5.29

If  $\Delta \vdash \mathbf{t} :: \kappa, \delta \in \mathcal{D}^{\uparrow}[\![\Delta]\!]^{\Sigma}, \Sigma_D; \Sigma_G, \mathbf{W}_k, \mathbf{W}_{\Gamma} \vdash \mathbf{E}^{\#} : \delta(\mathbf{t}) \Rightarrow s^{\dagger}$ , and  $(\forall \mathbf{u}, \mathbf{W}'. \mathbf{W}' \supseteq \mathbf{W} \wedge \Sigma_D; \Sigma_G, \mathbf{W}'_k, \mathbf{W}'_{\Gamma} \vdash \mathbf{u} : \delta(\mathbf{t}) \Rightarrow \exists e.\Sigma; \mathbf{W}'_k; \mathbf{W}'_{\Gamma} \vdash \mathbf{E}^{\#}[\mathbf{u}] : s^{\dagger} \uparrow e)$  then  $(\mathbf{W}, \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\![\mathbf{t}, s^{\dagger}]\!]_{\delta}^{\Sigma}$ .

**Proof**

To show  $(\mathbf{W}, \mathbf{E}^\#) \in \mathcal{K}^\uparrow[[\mathbf{t}, \mathbf{s}^\dagger]]_\delta^\Sigma$  it suffices to show for arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{m}) \in \mathcal{V}^\uparrow[[\mathbf{t}]]_\delta^\Sigma$ , that  $(\mathbf{W}', \mathbf{E}^\#[\mathbf{m}]) \in \mathcal{O}^\uparrow[[\mathbf{s}^\dagger]]_\delta^\Sigma$ .

By the definition of  $\mathcal{O}^\uparrow[[\mathbf{s}^\dagger]]_\delta^\Sigma$ , it suffices to show  $\exists e. \Sigma; \mathbf{W}'_k; \mathbf{W}'_\Gamma \vdash \mathbf{E}^\#[\mathbf{m}] : \mathbf{s}^{\dagger\uparrow} e$ , which follows by instantiating  $(\forall \mathbf{u}, \mathbf{W}'. \mathbf{W}' \supseteq \mathbf{W} \wedge \Sigma_D; \Sigma_G, \mathbf{W}'_k, \mathbf{W}'_\Gamma \vdash \mathbf{u} : \delta(\mathbf{t}) \implies \exists e. \Sigma; \mathbf{W}'_k; \mathbf{W}'_\Gamma \vdash \mathbf{E}^\#[\mathbf{u}] : \mathbf{s}^{\dagger\uparrow} e)$  with  $\mathbf{W}' \supseteq \mathbf{W}$ , and  $\mathbf{m}$ .  $\square$

**Lemma 5.30 (Relation equivalence is transitive)**

Let  $(\mathbf{t}, \mathbf{R}_1) \in \text{Rel}^\uparrow_{\kappa}^\Sigma$ ,  $(\mathbf{t}, \mathbf{R}) \in \text{Rel}^\uparrow_{\kappa}^\Sigma$ , and  $(\mathbf{t}, \mathbf{R}) \in \text{Rel}^\uparrow_{\kappa}^\Sigma$ .  
If  $\mathbf{R}_1 \equiv_{\kappa}^\Sigma \mathbf{R}$  and  $\mathbf{R} \equiv_{\kappa}^\Sigma \mathbf{R}_2$  then  $\mathbf{R}_1 \equiv_{\kappa}^{\text{D}; \text{G}} \mathbf{R}_2$ .

**Proof**

By induction on  $\kappa$ .

**Case  $\kappa = *$**  We must show  $\mathbf{R}_1 \equiv_{*}^\Sigma \mathbf{R}_2$ .

- For arbitrary  $(\mathbf{W}_1, \mathbf{m}_1) \in \mathbf{R}_1$ , we must show  $(\mathbf{W}_1, \mathbf{m}_1) \in \mathbf{R}_2$ .  
From  $\mathbf{R}_1 \equiv_{*}^\Sigma \mathbf{R}$ , we know  $(\mathbf{W}_1, \mathbf{m}) \in \mathbf{R}$ .  
From  $\mathbf{R} \equiv_{\kappa}^\Sigma \mathbf{R}_2$ , we know  $(\mathbf{W}_1, \mathbf{m}_1) \in \mathbf{R}_2$ .
- For arbitrary  $(\mathbf{W}_1, \mathbf{m}_1) \in \mathbf{R}_2$ , we must show  $(\mathbf{W}_1, \mathbf{m}_1) \in \mathbf{R}_1$ .  
This case is exactly symmetric to the previous case

**Case  $\kappa = \kappa_1 \rightarrow \kappa_2$ .**

For arbitrary  $(\mathbf{t}', \mathbf{R}') \in \text{Rel}^\uparrow_{\kappa_1}^\Sigma$ , we must show  $\mathbf{R}_1(\mathbf{t}', \mathbf{R}') \equiv_{\kappa_2}^\Sigma \mathbf{R}_2(\mathbf{t}', \mathbf{R}')$ .  
From  $\mathbf{R}_1 \equiv_{\kappa_1 \rightarrow \kappa_2}^\Sigma \mathbf{R}$ , we know  $\mathbf{R}_1(\mathbf{t}', \mathbf{R}') \equiv_{\kappa_2}^\Sigma \mathbf{R}(\mathbf{t}', \mathbf{R}')$ .  
From  $\mathbf{R} \equiv_{\kappa}^\Sigma \mathbf{R}_2$ , we know  $\mathbf{R}(\mathbf{t}', \mathbf{R}') \equiv_{\kappa_2}^\Sigma \mathbf{R}_2(\mathbf{t}', \mathbf{R}')$ .  
The result follows by IH applied to  $\kappa_2$ .  $\square$

**Lemma 5.31 (Back-translation type interpretation is well-formed)**

If  $\Delta \vdash \mathbf{t} :: \kappa$  and  $\delta \in \mathcal{D}^\uparrow[[\Delta]]^\Sigma$  then

1.  $(\delta(\mathbf{t}), \mathcal{T}^\uparrow[[\mathbf{t} :: \kappa]]_\delta^\Sigma) \in \text{Rel}^\uparrow_{\kappa}^\Sigma$
2. If  $\delta' \in \mathcal{D}^\uparrow[[\Delta]]^\Sigma$  such that  $\delta \equiv^\Sigma \delta'$ , then  $\mathcal{T}^\uparrow[[\mathbf{t} :: \kappa]]_\delta^\Sigma \equiv_{\kappa}^\Sigma \mathcal{T}^\uparrow[[\mathbf{t} :: \kappa]]_{\delta'}^\Sigma$

**Proof**

By induction on the derivation  $\Delta \vdash \mathbf{t} :: \kappa$ .

$$\text{(FK-Var)} \frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa}$$

1. We must show  $\delta_{\mathbf{R}}(\alpha) \in \text{Rel}^\uparrow_{\kappa}^\Sigma$ .  
Follows by assumption that  $\delta \in \mathcal{D}^\uparrow[[\Delta]]^\Sigma$ .
2. We must show  $\delta_{\mathbf{R}}(\alpha) \equiv_{\kappa}^\Sigma \delta'_{\mathbf{R}}(\alpha)$ .  
Follows by assumption that  $\delta \equiv^\Sigma \delta'$ .

$$\text{(FK-Unit)} \frac{}{\Delta \vdash \mathbf{1} :: *}$$

1. We must show  $\mathcal{E}^\uparrow[[\mathbf{1}]_\delta^\Sigma] \in \text{Rel}^\uparrow \Sigma_*$ .  
Follows by definition of  $\mathcal{E}^\uparrow[[s^\dagger]_\delta^\Sigma]$ .
2. We must show  $\mathcal{E}^\uparrow[[\mathbf{1}]_\delta^\Sigma] \equiv_*^\Sigma \mathcal{E}^\uparrow[[\mathbf{1}]_{\delta'}^\Sigma]$ , which follows trivially since back-translation does not rely on  $\delta$ .

$$\text{(FK-Pair)} \frac{\Delta \vdash \mathbf{t}_1 :: * \quad \Delta \vdash \mathbf{t}_2 :: *}{\Delta \vdash \mathbf{t}_1 \times \mathbf{t}_2 :: *}$$

**Case**  $\exists s_1, s_2. \mathbf{t}_1 = s_1^\dagger$ , and  $\mathbf{t}_2 = s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger \times s_2^\dagger]_\delta^\Sigma] \in \text{Rel}^\uparrow \Sigma_*$ .  
Follows by definition of  $\mathcal{E}^\uparrow[[s^\dagger]_\delta^\Sigma]$ .
2. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger \times s_2^\dagger]_\delta^\Sigma] \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger \times s_2^\dagger]_{\delta'}^\Sigma]$ , which follows trivially since back-translation does not rely on  $\delta$ .

This subcase follows by exactly the same reasoning in all kind  $*$  cases of proof, but it is repeated in detail anyway for completeness.

**Case**  $\exists s_1. \mathbf{t}_1 = s_1^\dagger$ , but  $\nexists s_2. \mathbf{t}_2 = s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma] \in \text{Rel}^\uparrow \Sigma_*$ .
  - For arbitrary  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$ ,  $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$ , we must show that  $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$ , which follows from Lemma 5.26 (Monotonicity of  $\mathcal{E}^\uparrow[[\mathbf{t}]_\delta^\Sigma]$ ), and
  - For arbitrary  $\mathbf{E}^\neq, \mathbf{t}$ , suppose  $(\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_1)_k, (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq : \delta(s_1^\dagger \times \mathbf{t}_2) \Rightarrow s'^\dagger$ , and  
 $\forall \mathbf{W}'_2, \mathbf{u}. \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \wedge \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_2)_k, (\mathbf{W}'_2)_\Gamma \vdash \mathbf{u} : \mathbf{t} \Rightarrow \exists e. \Sigma; (\mathbf{W}'_2)_k; (\mathbf{W}'_2)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{u}] : s'^\dagger \uparrow e$ .

We must show  $\exists e'. \Sigma; (\mathbf{W}'_1)_k; (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{m}] : s'^\dagger \uparrow e'$ .

Instantiating  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$  with  $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$ , it suffices to show  $(\mathbf{W}'_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_\delta^\Sigma]$ , which follows from Lemma 5.29.

This subcase follows by exactly the same reasoning in all kind  $*$  cases of proof, but it is repeated in detail anyway for completeness.

2. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma] \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_{\delta'}^\Sigma]$ .
  - If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_{\delta'}^\Sigma]$ .  
For arbitrary  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_\delta^\Sigma]$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\neq[\mathbf{m}]) \in \mathcal{O}^\uparrow[[s^\dagger]_{\delta'}^\Sigma]$ .  
By  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$  and the definition of  $\mathcal{O}^\uparrow[[s^\dagger]_{\delta'}^\Sigma]$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_\delta^\Sigma]$ .  
For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$ , we must show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s^\dagger]_{\delta'}^\Sigma]$ .  
Since  $\delta'$  is irrelevant to back-translation, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s^\dagger]_{\delta'}^\Sigma]$ .  
By Lemma 5.27 applied to  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_\delta^\Sigma]$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_{\delta'}^\Sigma]$ .  
By  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger \times \mathbf{t}_2, s^\dagger]_{\delta'}^\Sigma]$  it suffices to show  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_{\delta'}^\Sigma]$ .  
Since  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$ , it suffices to show  $\mathcal{E}^\uparrow[[s_1^\dagger]_\delta^\Sigma] \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger]_{\delta'}^\Sigma]$  and  $\mathcal{E}^\uparrow[[\mathbf{t}_2]_\delta^\Sigma] \equiv_*^\Sigma \mathcal{E}^\uparrow[[\mathbf{t}_2]_{\delta'}^\Sigma]$ , which follow from the IH part 2 applied to  $\Delta \vdash s_1^\dagger :: *$  and  $\Delta \vdash \mathbf{t}_2 :: *$ .
  - If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_{\delta'}^\Sigma]$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger \times \mathbf{t}_2]_\delta^\Sigma]$ .  
This direction is exactly symmetric to the previous direction.

**Case**  $\exists s_2.t_2 = s_2^\dagger$ , but  $\nexists s_1.t_1 = s_1^\dagger$ .

This case is symmetric to the previous case.

**Case**  $\nexists s_1.t_1 = s_1^\dagger$  and  $\exists s_2.t_2 = s_2^\dagger$ .

This case is identical to the non-translation subcases of the previous two cases.

$$\text{(FK-Sum)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *}$$

**Case**  $\exists s_1, s_2.t_1 = s_1^\dagger$ , and  $t_2 = s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger + s_2^\dagger]]_\delta^\Sigma \in \text{Rel}^\uparrow \Sigma_*$ .

Follows by definition of  $\mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

2. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger + s_2^\dagger]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger + s_2^\dagger]]_{\delta'}^\Sigma$ , which follows trivially since back-translation does not rely on  $\delta$ .

**Case**  $\exists s_1.t_1 = s_1^\dagger$ , but  $\nexists s_2.t_2 = s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma \in \text{Rel}^\uparrow \Sigma_*$ .

- For arbitrary  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ ,  $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$ , we must show that  $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , which follows from Lemma 5.26 (Monotonicity of  $\mathcal{E}^\uparrow[[t]]_\delta^\Sigma$ ), and

- For arbitrary  $\mathbf{E}^\neq, \mathbf{t}$ , suppose  $(\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_1)_k, (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq : \delta(s_1^\dagger + t_2) \Rightarrow s'^\dagger$ , and  $\forall \mathbf{W}'_2, \mathbf{u}. \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \wedge \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_2)_k, (\mathbf{W}'_2)_\Gamma \vdash \mathbf{u} : \mathbf{t} \Rightarrow \exists e. \Sigma; (\mathbf{W}'_2)_k; (\mathbf{W}'_2)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{u}] : s'^\dagger \uparrow e$ .

We must show  $\exists e'. \Sigma; (\mathbf{W}'_1)_k; (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{m}] : s'^\dagger \uparrow e'$ .

Instantiating  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$  with  $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$ , it suffices to show  $(\mathbf{W}'_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_\delta^\Sigma$ , which follows from Lemma 5.29.

2. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_{\delta'}^\Sigma$ .

- If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_{\delta'}^\Sigma$ .

For arbitrary  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\neq[\mathbf{m}]) \in \mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$  and the definition of  $\mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , we must show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

From  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , we know  $\mathbf{m}' = \text{inj}_i \mathbf{m}''$  and  $(\mathbf{W}_2, \mathbf{m}'') \in \mathcal{E}^\uparrow[[t_i]]_\delta^\Sigma$ .

Since  $\delta'$  is irrelevant to back-translation, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s^\dagger]]_{\delta'}^\Sigma$ .

By Lemma 5.27 applied to  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_\delta^\Sigma$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_{\delta'}^\Sigma$ .

By  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_{\delta'}^\Sigma$  it suffices to show  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger + t_2]]_{\delta'}^\Sigma$ .

Since  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , it suffices to show  $\mathcal{E}^\uparrow[[s_1^\dagger]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger]]_{\delta'}^\Sigma$  or  $\mathcal{E}^\uparrow[[t_2]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[t_2]]_{\delta'}^\Sigma$ , which follow from the IH part 2 applied to  $\Delta \vdash s_1^\dagger :: *$  and  $\Delta \vdash t_2 :: *$ .

- If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_{\delta'}^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ .

This direction is exactly symmetric to the previous direction.

**Case**  $\exists s_2.t_2 = s_2^\dagger$ , but  $\nexists s_1.t_1 = s_1^\dagger$ .

This case is symmetric to the previous case.

Case  $\not\exists s_1.t_1 = s_1^\dagger$  and  $\not\exists s_2.t_2 = s_2^\dagger$ .

This case is identical to the non-translation subcases of the previous two cases.

$$\text{(FK-Abs)} \frac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha :: \kappa. t :: *}$$

Case  $\exists s. \forall \alpha :: \kappa. t = T_\ell s^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow \llbracket T_\ell s^\dagger \rrbracket_\delta^\Sigma \in Rel^\uparrow \Sigma^*$ .

Follows by definition of  $\mathcal{E}^\uparrow \llbracket s^\dagger \rrbracket_\delta^\Sigma$ .

2. We must show  $\mathcal{E}^\uparrow \llbracket T_\ell s^\dagger \rrbracket_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow \llbracket T_\ell s^\dagger \rrbracket_\delta^\Sigma$ , which follows trivially since back-translation does not rely on  $\delta$ .

Case  $\not\exists s. \forall \alpha :: \kappa. t = T_\ell s^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma \in Rel^\uparrow \Sigma^*$ .

- For arbitrary  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall t :: \alpha. \rrbracket_\delta^\Sigma$ ,  $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$ , we must show that  $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$ , which follows from Lemma 5.26 (Monotonicity of  $\mathcal{E}^\uparrow \llbracket t \rrbracket_\delta^\Sigma$ ), and

- For arbitrary  $\mathbf{E}^\#$ ,  $\mathbf{t}$ , suppose  $(\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_1)_k, (\mathbf{W}'_1)_\Gamma) \vdash \mathbf{E}^\# : \delta(\forall \alpha :: \kappa. t) \Rightarrow s^\dagger$ , and

$$\forall \mathbf{W}'_2, \mathbf{u}. \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \wedge \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_2)_k, (\mathbf{W}'_2)_\Gamma \vdash \mathbf{u} : \mathbf{t} \implies \exists e. \Sigma; (\mathbf{W}'_2)_k; (\mathbf{W}'_2)_\Gamma \vdash \mathbf{E}^\#[\mathbf{u}] : s^\dagger \uparrow e.$$

We must show  $\exists e'. \Sigma; (\mathbf{W}'_1)_k; (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\#[\mathbf{m}] : s^\dagger \uparrow e'$ .

Instantiating  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$  with  $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$ , it suffices to show  $(\mathbf{W}'_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_\delta^\Sigma$ , which follows from Lemma 5.29.

2. We must show  $\mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$ .

- If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$ .

For arbitrary  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\#[\mathbf{m}]) \in \mathcal{O}^\uparrow \llbracket s^\dagger \rrbracket_\delta^\Sigma$ .

By  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$  and the definition of  $\mathcal{O}^\uparrow \llbracket s^\dagger \rrbracket_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$ , we must show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}']) \in \mathcal{O}^\uparrow \llbracket s^\dagger \rrbracket_\delta^\Sigma$ .

Since  $\delta'$  is irrelevant to back-translation, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}']) \in \mathcal{O}^\uparrow \llbracket s^\dagger \rrbracket_{\delta'}^\Sigma$ .

By Lemma 5.27 applied to  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_{\delta'}^\Sigma$  we conclude  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_{\delta'}^\Sigma$ .

By  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\uparrow \llbracket \forall \alpha :: \kappa. t, s^\dagger \rrbracket_{\delta'}^\Sigma$  it suffices to show  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_{\delta'}^\Sigma$ .

For arbitrary  $(\mathbf{t}', \mathbf{R}) \in Rel^\uparrow \Sigma_\kappa$ , it suffices to show  $\mathcal{E}^\uparrow \llbracket \mathbf{t} \rrbracket_{\delta[\alpha \mapsto (\mathbf{t}', \mathbf{R})]}^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow \llbracket \mathbf{t} \rrbracket_{\delta[\alpha \mapsto (\mathbf{t}', \mathbf{R})]}^\Sigma$  which follows from the IH part 2 applied to  $\Delta, \alpha :: \kappa \vdash \mathbf{t} :: *$ .

- If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_{\delta'}^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow \llbracket \forall \alpha :: \kappa. t \rrbracket_\delta^\Sigma$ .

This direction is exactly symmetric to the previous direction.

$$\text{(FK-Arrow)} \frac{\Delta \vdash t_1 :: * \quad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \rightarrow t_2 :: *}$$

Case  $\exists s_1, s_2. t_1 \rightarrow t_2 = s_1^\dagger \rightarrow s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow \llbracket s_1^\dagger \rightarrow s_2^\dagger \rrbracket_\delta^\Sigma \in Rel^\uparrow \Sigma^*$ .

Follows by definition of  $\mathcal{E}^\uparrow \llbracket s^\dagger \rrbracket_\delta^\Sigma$ .



2. We must show  $\mathcal{E}^\uparrow[[s_1^\dagger \rightarrow s_2^\dagger]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[s_1^\dagger \rightarrow s_2^\dagger]]_{\delta'}^\Sigma$ , which follows trivially since back-translation does not rely on  $\delta$ .

**Case**  $\not\exists s_1, s_2. t_1 \rightarrow t_2 = s_1^\dagger \rightarrow s_2^\dagger$ .

1. We must show  $\mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma \in Rel^\uparrow \Sigma_*$ .
  - For arbitrary  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$ ,  $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$ , we must show that  $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$ , which follows from Lemma 5.26 (Monotonicity of  $\mathcal{E}^\uparrow[[t]]_\delta^\Sigma$ ), and
  - For arbitrary  $\mathbf{E}^\neq, \mathbf{t}$ , suppose  $(\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_1)_k, (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq : \delta(t_1 \rightarrow t_2) \Rightarrow s'^\dagger$ , and  $\forall \mathbf{W}'_2, \mathbf{u}. \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \wedge \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, (\mathbf{W}'_2)_k, (\mathbf{W}'_2)_\Gamma \vdash \mathbf{u} : \mathbf{t} \Rightarrow \exists e. \Sigma; (\mathbf{W}'_2)_k; (\mathbf{W}'_2)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{u}] : s'^\dagger \uparrow e$ .  
We must show  $\exists e'. \Sigma; (\mathbf{W}'_1)_k; (\mathbf{W}'_1)_\Gamma \vdash \mathbf{E}^\neq[\mathbf{m}] : s'^\dagger \uparrow e'$ .  
Instantiating  $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$  with  $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$ , it suffices to show  $(\mathbf{W}'_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_\delta^\Sigma$ , which follows from Lemma 5.29.
2. We must show  $\mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_{\delta'}^\Sigma$ .
  - If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_{\delta'}^\Sigma$ .  
For arbitrary  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\neq[\mathbf{m}]) \in \mathcal{O}^\uparrow[[s'^\dagger]]_{\delta'}^\Sigma$ .  
By  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$  and the definition of  $\mathcal{O}^\uparrow[[s'^\dagger]]_{\delta'}^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_\delta^\Sigma$ .  
For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$ , we must show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s'^\dagger]]_{\delta'}^\Sigma$ .  
Since  $\delta'$  is irrelevant to back-translation, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\neq[\mathbf{m}']) \in \mathcal{O}^\uparrow[[s'^\dagger]]_\delta^\Sigma$ .  
By Lemma 5.27 applied to  $(\mathbf{W}_1, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_\delta^\Sigma$  we conclude  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_{\delta'}^\Sigma$ .  
By  $(\mathbf{W}_2, \mathbf{E}^\neq) \in \mathcal{K}^\uparrow[[t_1 \rightarrow t_2, s'^\dagger]]_{\delta'}^\Sigma$  it suffices to show  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[t_1 \rightarrow t_2]]_{\delta'}^\Sigma$ .  
Since  $(\mathbf{W}_2, \mathbf{m}') \in \mathcal{V}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$ , it suffices to show  $\mathcal{E}^\uparrow[[t_1]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[t_1]]_{\delta'}^\Sigma$  and  $\mathcal{E}^\uparrow[[t_2]]_\delta^\Sigma \equiv_*^\Sigma \mathcal{E}^\uparrow[[t_2]]_{\delta'}^\Sigma$ , which follow from the IH part 2 applied to  $\Delta \vdash t_1 :: *$  and  $\Delta \vdash t_2 :: *$ .
  - If  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_{\delta'}^\Sigma$  then  $(\mathbf{W}_1, \mathbf{m}) \in \mathcal{E}^\uparrow[[t_1 \rightarrow t_2]]_\delta^\Sigma$ .  
This direction is exactly symmetric to the previous direction.

(FK-Fun) 
$$\frac{\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2}{\Delta \vdash \lambda \alpha :: \kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2}$$

1. We must show  $(\delta(\lambda \alpha :: \kappa. \mathbf{t}), \mathcal{T}^\uparrow[[\lambda \alpha :: \kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma) \in Rel^\uparrow \Sigma_{\kappa_1 \rightarrow \kappa_2}$ .  
For arbitrary  $(\mathbf{t}_1, \mathbf{R}_1) \in Rel^\uparrow \Sigma_{\kappa_1}$ , we must show
  - (a)  $(\mathbf{t}[\mathbf{t}_1/\alpha], \mathcal{T}^\uparrow[[\mathbf{t} :: \kappa_2]]_{\delta[\alpha \mapsto (\mathbf{t}_1, \mathbf{R}_1)]}^\Sigma) \in Rel^\uparrow \Sigma_{\kappa_2}$  which follows from IH part 1 applied to  $\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$ , and
  - (b) For arbitrary  $(\mathbf{t}'_1, \mathbf{R}'_1) \in Rel^\uparrow \Sigma_{\kappa_1}$ , if  $(\mathbf{t}_1, \mathbf{R}_1) \equiv_{\kappa_1}^\Sigma (\mathbf{t}'_1, \mathbf{R}'_1)$  then  $\mathcal{T}^\uparrow[[\mathbf{t} :: \kappa_2]]_{\delta[\alpha \mapsto (\mathbf{t}_1, \mathbf{R}_1)]}^\Sigma \equiv_{\kappa_2}^\Sigma \mathcal{T}^\uparrow[[\mathbf{t} :: \kappa_2]]_{\delta[\alpha \mapsto (\mathbf{t}'_1, \mathbf{R}'_1)]}^\Sigma$ , which follows from IH part 2 applied to  $\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$ .
2. We must show  $\mathcal{T}^\uparrow[[\lambda \alpha :: \kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa_1 \rightarrow \kappa_2}^\Sigma \mathcal{T}^\uparrow[[\lambda \alpha :: \kappa. \mathbf{t} :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$ .  
This follows from IH part 2 applied to  $\Delta, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$ .

(FK-App) 
$$\frac{\Delta \vdash \mathbf{t}_1 :: \kappa_1 \rightarrow \kappa_2 \quad \Delta \vdash \mathbf{t}_2 :: \kappa_1}{\Delta \vdash \mathbf{t}_1 \mathbf{t}_2 :: \kappa_2}$$

1. We must show  $(\delta(t_1 t_2), \mathcal{T}^\uparrow[[t_1 t_2 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma) \in Rel^\uparrow_{\kappa}^\Sigma$ .

By IH part 1 applied to  $\Delta \vdash t_1 :: \kappa_1 \rightarrow \kappa_2$  we know  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \in Rel^\uparrow_{\kappa_1 \rightarrow \kappa_2}^\Sigma$ . By definition of  $Rel^\uparrow_{\kappa_1 \rightarrow \kappa_2}^\Sigma$  it suffices to show  $(\delta(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_\delta^\Sigma) \in Rel^\uparrow_{\kappa_1}^\Sigma$ , which follows by IH part 1 applied to  $\Delta \vdash t_2 :: \kappa_1$ .

2. We must show  $\mathcal{T}^\uparrow[[t_1 t_2 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa}^\Sigma \mathcal{T}^\uparrow[[t_1 t_2 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$

It suffices to show

$$(\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma (\delta(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_\delta^\Sigma)) \equiv_{\kappa_2}^\Sigma (\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma (\delta'(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_{\delta'}^\Sigma)).$$

By IH part 2 we know  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa_1 \rightarrow \kappa_2}^\Sigma \mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$  and

$$\mathcal{T}^\uparrow[[t_2 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa_1}^\Sigma \mathcal{T}^\uparrow[[t_2 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma,$$

By IH part 1 we know  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma \in Rel^\uparrow_{\kappa_1 \rightarrow \kappa_2}^\Sigma$  and  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \in Rel^\uparrow_{\kappa_1 \rightarrow \kappa_2}^\Sigma$ .

Instantiating  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa_1 \rightarrow \kappa_2}^\Sigma \mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$  with  $\mathcal{T}^\uparrow[[t_2 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$  we know

$$(\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma (\delta(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_\delta^\Sigma)) \equiv_{\kappa_2}^\Sigma (\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma (\delta(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_\delta^\Sigma)).$$

Instantiating  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma \in Rel^\uparrow_{\kappa_1 \rightarrow \kappa_2}^\Sigma$  with  $\mathcal{T}^\uparrow[[t_2 :: \kappa_1 \rightarrow \kappa_2]]_\delta^\Sigma \equiv_{\kappa_1}^\Sigma \mathcal{T}^\uparrow[[t_2 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma$ , we know

$$(\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma (\delta(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_\delta^\Sigma)) \equiv_{\kappa_2}^\Sigma (\mathcal{T}^\uparrow[[t_1 :: \kappa_1 \rightarrow \kappa_2]]_{\delta'}^\Sigma (\delta'(t_2), \mathcal{T}^\uparrow[[t_2 :: \kappa_1]]_{\delta'}^\Sigma)).$$

The result follows by Lemma 5.30 (Relation equivalence is transitive).

□

### Lemma 5.32 (Back-translation compositionality)

If  $\Delta, \alpha :: \kappa_2 \vdash t_1 :: \kappa_2$ ,  $\Delta \vdash t_2 :: \kappa_2$ , and  $R_2 = \mathcal{T}^\uparrow[[t_2 :: \kappa_2]]_\delta^\Sigma$ , then  $\mathcal{T}^\uparrow[[t_1 :: \kappa_1]]_{\delta[\alpha \mapsto (t_2, R_2)]}^\Sigma \equiv_{\kappa_1}^\Sigma \mathcal{T}^\uparrow[[t_1[\alpha/t_2] :: \kappa_1]]_\delta^\Sigma$ .

### Theorem 5.33 (Coherence for back-translation relation)

If  $\Delta \vdash t_1 :: \kappa$ ,  $\Delta \vdash t_2 :: \kappa$ ,  $t_1 \equiv t_2$ , and  $\delta \in \mathcal{D}^\uparrow[[\Delta]]^\Sigma$ , then  $\mathcal{T}^\uparrow[[t_1 :: \kappa]]_\delta^\Sigma \equiv_{\kappa}^\Sigma \mathcal{T}^\uparrow[[t_2 :: \kappa]]_\delta^\Sigma$ .

### Lemma 5.34 (Fundamental Property of Back-translation Logical Relation)

If  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, G_k, \Gamma \vdash m : t$  then  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, G_k, \Gamma \vDash^\dagger m : t$

*Proof idea:* Proof by induction on the typing derivation  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, G_k, \Gamma \vdash m : t$ . Most cases will have subcases for either  $\exists s.t = s^\dagger$  or not.

For brevity,  $\mathbf{W}$  for  $(G_k; \Gamma^\dagger)$  unless the world requires particular attention.

**Proof**

**Case FT-Unit**

We know

$$\text{(FT-Unit)} \frac{}{D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash \langle \rangle : 1}$$

We must show  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vDash^\dagger \langle \rangle : 1$ .

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\uparrow[[\Delta]]^\Sigma$ ,  $((G_k; \Gamma_2^\dagger), \gamma) \in \mathcal{G}^\uparrow[[\Gamma_1^\dagger]]_\delta^\Sigma$ ,  $((G_k; \Gamma_2^\dagger), \gamma) \in \mathcal{G}^\uparrow[[\Gamma]_\delta^\Sigma]$ , we must show

$$((G_k; \Gamma_2^\dagger), \langle \rangle) \in \mathcal{E}^\uparrow[[1]]_\delta^\Sigma, \text{ which follows by FD-Unit.}$$

**Case FT-Var**

We know

$$\frac{\mathbf{x} : \mathbf{t} \in \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{G}_k, \Gamma}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{x} : \mathbf{t}}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\dagger \mathbf{x} : \mathbf{t}$ .

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,  $((\mathbf{G}_k; \Gamma_2^\dagger), \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$ ,  $((\mathbf{G}_k; \Gamma_2^\dagger), \gamma) \in \mathcal{G}^\dagger[\Gamma]^\Sigma$ , we must show

$$\delta(\gamma(\gamma(\mathbf{x}))) \in \mathcal{E}^\dagger[\mathbf{t}]^\Sigma.$$

For brevity, let  $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{x})))$ .

**Case  $\mathbf{x} : \mathbf{t} \in \Gamma$**

Then we know  $\exists \mathbf{s}. \mathbf{t} = \mathbf{s}^\dagger$ , and  $\mathbf{x}$  is in the domain of  $\gamma$ . Let  $\gamma(\mathbf{x}) = \mathbf{m}$ . We must show

$$((\mathbf{G}_k; \Gamma_2^\dagger), \mathbf{m}) \in \mathcal{E}^\dagger[\mathbf{t}]^\Sigma$$

By definition of  $\mathcal{G}^\dagger[\Gamma]^\Sigma$  and by assumption that  $((\mathbf{G}_k; \Gamma_2^\dagger), \gamma) \in \mathcal{G}^\dagger[\Gamma]^\Sigma$ , we know  $((\mathbf{G}_k; \Gamma_2^\dagger), \mathbf{m}) \in \mathcal{E}^\dagger[\mathbf{t}]^\Sigma$

**Case  $\mathbf{x} : \mathbf{t} \in \Gamma_1^\dagger$**

Then we know  $\exists \mathbf{s}. \mathbf{t} = \mathbf{s}^\dagger$  and  $\mathbf{x}$  is in the domain of  $\gamma$ . Let  $\gamma(\mathbf{x}) = \mathbf{m}$ . We must show

$$(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^\dagger[\mathbf{s}^\dagger]^\Sigma.$$

By definition of  $\mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$  and by assumption that  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$ , we know  $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^\dagger[\mathbf{s}^\dagger]^\Sigma$ .

**Case  $\mathbf{x} : \mathbf{t} \in \mathbf{G}_k$**

Then we know  $\mathbf{x} = \mathbf{k}$ ,  $\mathbf{t} = \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell \mathbf{s})^\dagger$  and  $\mathbf{x}$  is not in the domain of  $\gamma$  or  $\gamma$ .

We must show  $((\mathbf{G}_k; \Gamma_2^\dagger), \mathbf{k}) \in \mathcal{E}^\dagger[\mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell \mathbf{s})^\dagger]^\Sigma$ , which follows from FD-K.

**Case  $\mathbf{x} : \mathbf{t} \in \Gamma_2^\dagger$**

Then we know  $\mathbf{x} : \mathbf{t} \in \Gamma_2^\dagger$ , and  $\mathbf{t} = \mathbf{s}^\dagger$  and  $\mathbf{x}$  is not in the domain of  $\gamma$  or  $\gamma$ .

$((\mathbf{G}_k; \Gamma_2^\dagger), \mathbf{x}) \in \mathcal{E}^\dagger[\mathbf{s}^\dagger]^\Sigma$  follows by FD-Var.

**Case  $\mathbf{x} : \mathbf{t} \in \mathbf{G}_\ell$**

Then we know  $\exists \ell. \mathbf{t} = \hat{\alpha}_\ell$  and  $\mathbf{x}$  is not in the domain of  $\gamma$  or  $\gamma$ . We must show  $(\mathbf{W}, \mathbf{x}) \in$

$$\mathcal{E}^\dagger[\hat{\alpha}_\ell]^\Sigma.$$

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{E}^\#) \in \mathcal{K}^\dagger[\hat{\alpha}_\ell, \mathbf{s}^\dagger]^\Sigma$ , it suffices to show  $(\mathbf{W}', \mathbf{E}^\#[\mathbf{x}]) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]^\Sigma$ .

There is only one  $\mathbf{E}^\#$  in which a  $\mathbf{x} : \hat{\alpha}_\ell$  can appear:  $[\cdot]_{\mathbf{T}}$ . However, the result type cannot be  $\mathbf{s}^\dagger$ . So, since there is no well-typed context of type  $(\hat{\alpha}_\ell) \Rightarrow \mathbf{s}^\dagger$ , the statement  $\forall (\mathbf{W}', \mathbf{E}^\#) \in$

$\mathcal{K}^\dagger[\hat{\alpha}_\ell, \mathbf{s}^\dagger]^\Sigma. (\mathbf{W}', \mathbf{E}^\#[\mathbf{x}]) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]^\Sigma$  holds vacuously. Therefore  $(\mathbf{W}', \mathbf{x}) \in \mathcal{E}^\dagger[\hat{\alpha}_\ell]^\Sigma$

Note the implication here is that  $\mathbf{x}$  can appear in a term, but never in evaluation position.

It can be, for instance, passed as an argument to a function that never its argument other than by passing it to another function that behaves similarly. The back-translation reduction will cause such a term to be discarded. So  $\mathbf{x}$  exists only as a proof object to allow terms corresponding to protected source terms to be well-typed.

**Case  $\mathbf{x} : \mathbf{t} \in \mathbf{G}_\leq$**

The argument for this case is similar to the previous case. We know  $\mathbf{x}$  is one of the proof constructors  $\hat{\mathbf{P}}_1, \hat{\mathbf{P}}_\times, \hat{\mathbf{P}}_{\rightarrow}, \hat{\mathbf{P}}_{\mathbf{T}_1}$ , or  $\hat{\mathbf{P}}_{\mathbf{T}_2}$ .

Formally, we know  $\exists \mathbf{s}. \mathbf{t} = \mathbf{s}^\dagger$  and  $\mathbf{x}$  is not in the domain of  $\gamma$  or  $\gamma$ . We must show  $(\mathbf{W}, \mathbf{x}) \in$

$$\mathcal{E}^\dagger[\mathbf{t}]^\Sigma.$$

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{E}^\#) \in \mathcal{K}^\dagger[\mathbf{t}, \mathbf{s}^\dagger]^\Sigma$ , we must show  $(\mathbf{W}', \mathbf{E}^\#[\mathbf{x}]) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]^\Sigma$ .

However, by inspecting the types of  $\mathbf{x}$  and of each  $\mathbf{E}^\#$ , we know no such  $\mathbf{E}^\#$  exists.

Consider for instance  $\mathbf{x} = \hat{\mathbf{p}}_{\times}$ . We know:

$$\mathbf{t} = \forall \beta_{\ell}::*. \forall \alpha_1::*. \forall \alpha_2::*. ((\hat{\alpha}_{\leq} \beta_{\ell} \alpha_1) \times (\hat{\alpha}_{\leq} \beta_{\ell} \alpha_2)) \rightarrow (\hat{\alpha}_{\leq} \beta_{\ell} (\alpha_1 \times \alpha_2))$$

A term of this type can appear in an evaluation context such as  $[\cdot]_{\mathbf{T}}[\hat{\alpha}_{\ell}][t_1][t_2]\mathbf{m}'$ . However, the result of this evaluation context is not of translation type, and cannot appear in any other  $\mathbf{E}^{\#}$  in evaluation position. For every possible  $\mathbf{x} : \mathbf{t} \in \mathbf{G}_{\leq}$ , the same is true of all  $\mathbf{E}^{\#}$  in which  $\mathbf{x}$  can appear.

Since there is no  $\mathbf{E}^{\#}$  that results in a translation type,  $(\mathbf{W}', \mathbf{x}) \in \mathcal{E}^{\uparrow}[\mathbf{t}]_{\delta}^{\Sigma}$  is vacuously true.

### Case FT-Pair

We know

$$\frac{\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vdash \mathbf{m}_1 : \mathbf{t}_1 \quad \mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vdash \mathbf{m}_2 : \mathbf{t}_2}{\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vdash \langle \mathbf{m}_1, \mathbf{m}_2 \rangle : \mathbf{t}_1 \times \mathbf{t}_2}$$

We must show  $\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vDash^{\uparrow} \langle \mathbf{m}_1, \mathbf{m}_2 \rangle : \mathbf{t}_1 \times \mathbf{t}_2$ .

For arbitrary  $\Gamma_1^{\dagger}, \Gamma_2^{\dagger}, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^{\uparrow}[\Delta]_{\delta}^{\Sigma}$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^{\uparrow}[\Gamma_1^{\dagger}]_{\delta}^{\Sigma}$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^{\uparrow}[\Gamma]_{\delta}^{\Sigma}$ , we must show

$$(\mathbf{W}, \delta(\gamma(\gamma(\langle \mathbf{m}_1, \mathbf{m}_2 \rangle)))) \in \mathcal{E}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2]_{\delta}^{\Sigma}.$$

For brevity, let  $\mathbf{m}'_1 = \delta(\gamma(\gamma(\mathbf{m}_1)))$ , and  $\mathbf{m}'_2 = \delta(\gamma(\gamma(\mathbf{m}_2)))$ .

**Case**  $\exists s_1, s_2. \mathbf{t}_1 \times \mathbf{t}_2 = s_1^{\dagger} \times s_2^{\dagger}$

We must show  $(\mathbf{W}, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{E}^{\uparrow}[s_1^{\dagger} \times s_2^{\dagger}]_{\delta}^{\Sigma}$ .

By the IH applied to  $\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vdash \mathbf{m}_1 : \mathbf{t}_1$ , and  $\mathbf{D}_{\ell}, \Delta; \mathbf{G}_{\ell}, \mathbf{G}_{\leq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger}, \Gamma \vdash \mathbf{m}_2 : \mathbf{t}_2$ , we know:

$$(\mathbf{W}, \mathbf{m}'_1) \in \mathcal{E}^{\uparrow}[s_1^{\dagger}]_{\delta}^{\Sigma} \text{ and } (\mathbf{W}, \mathbf{m}'_2) \in \mathcal{E}^{\uparrow}[s_2^{\dagger}]_{\delta}^{\Sigma}.$$

By FD-Pair, we know  $(\mathbf{W}, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{E}^{\uparrow}[s_1^{\dagger} \times s_2^{\dagger}]_{\delta}^{\Sigma}$ .

**Case**  $\exists s_1. \mathbf{t}_1 = s_1^{\dagger}$ , but  $\nexists s_2. \mathbf{t}_2 = s_2^{\dagger}$ .

We must show  $(\mathbf{W}, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{E}^{\uparrow}[s_1^{\dagger} \times \mathbf{t}_2]_{\delta}^{\Sigma}$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[s_1^{\dagger} \times \mathbf{t}_2, s^{\dagger}]_{\delta}^{\Sigma}$ , we must show  $(\mathbf{W}', \mathbf{E}^{\#}[\langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle]) \in \mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma}$ .

By the definition of  $\mathcal{K}^{\uparrow}[s_1^{\dagger} \times \mathbf{t}_2, s^{\dagger}]_{\delta}^{\Sigma}$ , it suffices to show  $(\mathbf{W}', \langle \mathbf{m}_1, \mathbf{m}_2 \rangle) \in \mathcal{V}^{\uparrow}[s_1^{\dagger} \times \mathbf{t}_2]_{\delta}^{\Sigma}$ .

By the definition of  $\mathcal{V}^{\uparrow}[s_1^{\dagger} \times \mathbf{t}_2]_{\delta}^{\Sigma}$ , it suffices to show  $(\mathbf{W}', \mathbf{m}'_1) \in \mathcal{E}^{\uparrow}[s_1^{\dagger}]_{\delta}^{\Sigma}$  and  $(\mathbf{W}', \mathbf{m}'_2) \in \mathcal{E}^{\uparrow}[\mathbf{t}_2]_{\delta}^{\Sigma}$ , which follow by the IH applied to  $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger} \vdash \mathbf{m}_i : \mathbf{t}_i$  and Lemma 5.26.

**Case**  $\exists s_2. \mathbf{t}_2 = s_2^{\dagger}$ , but  $\nexists s_1. \mathbf{t}_1 = s_1^{\dagger}$ .

The proof is analogous to the previous case.

**Case**  $\nexists s_1. \mathbf{t}_1 = s_1^{\dagger}$  and  $\nexists s_2. \mathbf{t}_2 = s_2^{\dagger}$ .

The proof is analogous to the two previous cases. We repeat it since both types are non-translation types and it is not obviously symmetric.

We must show  $(\mathbf{W}, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{E}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2]_{\delta}^{\Sigma}$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{E}^{\#}) \in \mathcal{K}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2, s^{\dagger}]_{\delta}^{\Sigma}$ , it suffices to show  $(\mathbf{W}', \mathbf{E}^{\#}[\langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle]) \in \mathcal{O}^{\uparrow}[s^{\dagger}]_{\delta}^{\Sigma}$ .

By definition of  $\mathcal{K}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2, s^{\dagger}]_{\delta}^{\Sigma}$ , it suffices to show  $(\mathbf{W}', \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{V}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2]_{\delta}^{\Sigma}$ .

By definition of  $\mathcal{V}^{\uparrow}[\mathbf{t}_1 \times \mathbf{t}_2]_{\delta}^{\Sigma}$  it suffices to show  $(\mathbf{W}', \mathbf{m}'_i) \in \mathcal{E}^{\uparrow}[\mathbf{t}_i]_{\delta}^{\Sigma}$ , which follows by the IH applied to  $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\dagger} \vdash \mathbf{m}_i : \mathbf{t}_i$  and Lemma 5.26.

### Case FT-Prj

We know

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \text{prj}_1 \mathbf{m} : \mathbf{t}_i}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\dagger \text{prj}_1 \mathbf{m} : \mathbf{t}_i$ .

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma'$  such that  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma]^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\text{prj}_1 \mathbf{m}))) \in \mathcal{E}^\dagger[\mathbf{t}_i]^\Sigma.$$

For brevity, let  $\mathbf{m}' = \delta(\gamma(\mathbf{m}))$ .

We consider only the case when  $i = 1$  and  $\mathbf{t}_i = \mathbf{t}_1$ , as the other case follows symmetrically.

**Case**  $\exists s_1, s_2. \mathbf{t}_1 \times \mathbf{t}_2 = s_1^\dagger \times s_2^\dagger$

We must show  $(\mathbf{W}, \text{prj}_1 \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger]^\Sigma$ .

By FD-Subterm and FD-Prj, it suffices to show  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger \times s_2^\dagger]^\Sigma$ , which follows by the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$ .

**Case**  $\exists s_1. \mathbf{t}_1 = s_1^\dagger$ , but  $\nexists s_2. \mathbf{t}_2 = s_2^\dagger$ .

We must show  $(\mathbf{W}, \text{prj}_1 \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger]^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$ , we know  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger \times \mathbf{t}_2]^\Sigma$ .

By definition of  $\mathcal{E}^\dagger[s_1^\dagger \times \mathbf{t}_2]^\Sigma$ , it suffices to show  $(\mathbf{W}, \text{prj}_1 [\cdot]_{\mathbf{T}}) \in \mathcal{K}^\dagger[s_1^\dagger \times \mathbf{t}_2, \delta]^\Sigma$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{V}^\dagger[s_1^\dagger \times \mathbf{t}_2]^\Sigma$ , it suffices to show  $(\mathbf{W}', \text{prj}_1 \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{O}^\dagger[s_1^\dagger]^\Sigma$ .

Note that  $\text{prj}_1 \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle = \mathbf{E}^\#[\mathbf{u}]$ .

By FD-Val and the definition of  $\mathcal{E}^\dagger[s_1^\dagger]^\Sigma$ , it suffices to show  $(\mathbf{W}', \mathbf{m}'_1) \in \mathcal{E}^\dagger[s_1^\dagger]^\Sigma$ , which follows by assumption that  $(\mathbf{W}', \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{V}^\dagger[s_1^\dagger \times \mathbf{t}_2]^\Sigma$ .

**Case**  $\exists s_2. \mathbf{t}_2 = s_2^\dagger$ , but  $\nexists s_1. \mathbf{t}_1 = s_1^\dagger$ .

We must show  $(\mathbf{W}, \text{prj}_1 \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{t}_1]^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\dagger[\mathbf{t}_1, s_2^\dagger]^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\#[\text{prj}_1 \mathbf{m}']) \in \mathcal{O}^\dagger[s_2^\dagger]^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$ , we know  $(\mathbf{W}_1, \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{t}_1 \times s_2^\dagger]^\Sigma$ .

By  $(\mathbf{W}_1, \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{t}_1 \times s_2^\dagger]^\Sigma$ , to show  $(\mathbf{W}_1, \mathbf{E}^\#[\text{prj}_1 \mathbf{m}']) \in \mathcal{O}^\dagger[s_2^\dagger]^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\#[\text{prj}_1 [\cdot]_{\mathbf{T}}]) \in \mathcal{K}^\dagger[\mathbf{t}_1 \times s_2^\dagger, s_2^\dagger]^\Sigma$ .

Note that  $\text{prj}_1 [\cdot]_{\mathbf{T}} : (\mathbf{t}_1 \times s_2^\dagger) \Rightarrow \mathbf{t}_1$ , so  $\mathbf{E}^\#[\text{prj}_1 [\cdot]_{\mathbf{T}}] = \mathbf{E}_1^\#$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{V}^\dagger[\mathbf{t}_1 \times s_2^\dagger]^\Sigma$ , we must show  $(\mathbf{W}_2, \mathbf{E}^\#[\text{prj}_1 \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle]) \in \mathcal{O}^\dagger[s_2^\dagger]^\Sigma$ .

Note that  $\mathbf{E}^\#[\text{prj}_1 \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle] = \mathbf{E}_1^\#[\mathbf{u}]$ , so by FD-Val, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_1]) \in \mathcal{E}^\dagger[s_2^\dagger]^\Sigma$ .

From  $(\mathbf{W}_2, \langle \mathbf{m}'_1, \mathbf{m}'_2 \rangle) \in \mathcal{V}^\dagger[\mathbf{t}_1 \times s_2^\dagger]^\Sigma$  we know  $(\mathbf{W}_2, \mathbf{m}'_1) \in \mathcal{E}^\dagger[\mathbf{t}_1]^\Sigma$ . Instantiating the latter with  $\mathbf{E}^\#$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_1]) \in \mathcal{O}^\dagger[s_2^\dagger]^\Sigma$ .

**Case**  $\nexists s_1. \mathbf{t}_1 = s_1^\dagger$  and  $\nexists s_2. \mathbf{t}_2 = s_2^\dagger$ .

This case is analogous to the previous case. Note in the previous case, the fact that  $\mathbf{t}_2 = s_2^\dagger$  had no effect, since the type of the pair  $\mathbf{t}_1 \times s_2^\dagger$  and the type of the projection  $\mathbf{t}_1$  are not translation types.

**Case FT-Sum**

We know

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_i}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \text{inj}_i \mathbf{m} : t_1 + t_2}$$

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\uparrow[\Delta]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[\Gamma_1^\dagger]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[\Gamma]^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\gamma(\text{inj}_i \mathbf{m})))) \in \mathcal{E}^\uparrow[t_1 + t_2]^\Sigma.$$

For brevity, let  $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{m})))$ .

**Case**  $\exists s_1, s_2. t_1 + t_2 = s_1^\dagger + s_2^\dagger$ .

We must show  $(\mathbf{W}, \text{inj}_i \mathbf{m}') \in \mathcal{E}^\uparrow[s_1^\dagger + s_2^\dagger]^\Sigma$ .

By FD-Sum, it suffices to show  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\uparrow[s_i^\dagger]^\Sigma$ , which follows by the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_i$ .

**Case**  $\exists s_1. t_1 = s_1$ , but  $\nexists s_2. t_2 = s_2$ .

We must show  $(\mathbf{W}, \text{inj}_i \mathbf{m}') \in \mathcal{E}^\uparrow[s_1^\dagger + t_2]^\Sigma$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \mathbf{E}^\#) \in \mathcal{K}^\uparrow[s_1^\dagger + t_2, s^\dagger]^\Sigma$ , we must show  $(\mathbf{W}', \mathbf{E}^\#[\text{inj}_i \mathbf{m}']) \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ .

By  $(\mathbf{W}', \mathbf{E}^\#) \in \mathcal{K}^\uparrow[s_1^\dagger + t_2, s^\dagger]^\Sigma$ , it suffices to show  $(\mathbf{W}', \text{inj}_i \mathbf{m}') \in \mathcal{V}^\uparrow[s_1^\dagger + t_2]^\Sigma$ .

By definition of  $\mathcal{V}^\uparrow[s_1^\dagger + t_2]^\Sigma$ , it suffices to show  $(\mathbf{W}', \mathbf{m}') \in \mathcal{E}^\uparrow[t_i]^\Sigma$ , which follows by IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_i$  and Lemma 5.26.

**Case**  $\exists s_2. t_2 = s_2$ , but  $\nexists s_1. t_1 = s_1$ .

This case is exactly identical to the previous case. Note that in the previous case, we never use the fact that  $t'_i$  may be a translation type.

**Case**  $\nexists s_1. t_1 = s_1$ , and  $\nexists s_2. t_2 = s_2$ .

This case is exactly identical to the previous two cases. Note that in the previous cases, we never use the fact that  $t'_i$  may be a translation type.

**Case FT-Case**

We know

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_1 + t_2 \quad \mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : t_1, \Gamma \vdash \mathbf{m}_1 : t \quad \mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_2 : t_2, \Gamma \vdash \mathbf{m}_2 : t}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 \mathbf{x}_1. \mathbf{m}_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}_2 : t}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\uparrow \text{case } \mathbf{m} \text{ of } \text{inj}_1 \mathbf{x}_1. \mathbf{m}_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}_2 : t$

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\uparrow[\Delta]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[\Gamma_1^\dagger]^\Sigma$ ,

$(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[\Gamma]^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\gamma(\text{case } \mathbf{m} \text{ of } \text{inj}_1 \mathbf{x}_1. \mathbf{m}_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}_2)))) \in \mathcal{E}^\uparrow[t]^\Sigma.$$

For brevity,  $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{m})))$ ,  $\mathbf{m}'_1 = \delta(\gamma(\gamma(\mathbf{m}_1)))$ , and  $\mathbf{m}'_2 = \delta(\gamma(\gamma(\mathbf{m}_2)))$ .

**Case**  $\exists s. t = s^\dagger$

**Case**  $\exists s_1^\dagger, s_2^\dagger. t_1 + t_2 = s_1^\dagger + s_2^\dagger$

We must show  $(\mathbf{W}, \text{case } \mathbf{m}' \text{ of } \text{inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2) \in \mathcal{E}^\uparrow[s^\dagger]^\Sigma$

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_1 + t_2$ , we know  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\uparrow[s_1^\dagger + s_2^\dagger]^\Sigma$ .

Let  $\mathbf{W}' = (\mathbf{G}_k; \Gamma_2^\dagger, \mathbf{x} : s_1^\dagger)$ .

By Lemma 5.28 (Monotonicity of  $\mathcal{G}^\dagger[\Gamma]_\delta^\Sigma$ ), we know  $(\mathbf{W}', \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]_\delta^\Sigma$  and  $(\mathbf{W}', \gamma) \in \mathcal{E}^\dagger[\Gamma]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x} : s_1^\dagger, \Gamma \vdash \mathbf{m}_1 : \mathbf{t}$ , we know  $(\mathbf{W}', \mathbf{m}'_1) \in \mathcal{E}^\dagger[s_1^\dagger]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x} : s_2^\dagger, \Gamma \vdash \mathbf{m}_2 : \mathbf{t}$ , we know  $(\mathbf{W}', \mathbf{m}'_2) \in \mathcal{E}^\dagger[s_2^\dagger]_\delta^\Sigma$ .

By FD-Subterm and FD-Case, we conclude  $(\mathbf{W}, \text{case } \mathbf{m}' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

**Case**  $\exists s_1^\dagger. t_1 = s_1^\dagger$ , but  $\nexists s_2^\dagger. t_2 = s_2^\dagger$ .

We must show  $(\mathbf{W}, \text{case } \mathbf{m}' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t_1 + t_2$ , we know  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger + t_2]_\delta^\Sigma$ .

By  $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^\dagger[s_1^\dagger + t_2]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}, \text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2) \in \mathcal{K}^\dagger[s_1^\dagger + t_2, s^\dagger]_\delta^\Sigma$ .

Note that  $\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2 : s_1^\dagger + t_2 \Rightarrow s^\dagger$  is an  $\mathbf{E}^\#$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \text{inj}_i \mathbf{m}'') \in \mathcal{V}^\dagger[s_1^\dagger + t_2]_\delta^\Sigma$ , we must show

$(\mathbf{W}', \text{case } \text{inj}_i \mathbf{m}'' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2) \in \mathcal{O}^\dagger[s^\dagger]_\delta^\Sigma$ .

By FD-Val and definition of  $\mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}', \mathbf{m}'_i[\mathbf{m}''/x_i]) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

**Case**  $i = 1$

From  $(\mathbf{W}', \text{inj}_i \mathbf{m}'') \in \mathcal{V}^\dagger[s_1^\dagger + t_2]_\delta^\Sigma$  we know  $(\mathbf{W}', \mathbf{m}'') \in \mathcal{E}^\dagger[s_1^\dagger]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : s_1^\dagger, \Gamma \vdash \mathbf{m}_1 : s^\dagger$ , we know

$\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : s_1^\dagger, \Gamma \vDash^\dagger \mathbf{m}_1 : s^\dagger$ .

Recall that  $\mathbf{W}' \supseteq \mathbf{W}$ . By Lemma 5.28 (Monotonicity of  $\mathcal{G}^\dagger[\Gamma]_\delta^\Sigma$ ), we know  $(\mathbf{W}', \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]_\delta^\Sigma$  and  $(\mathbf{W}', \gamma) \in \mathcal{G}^\dagger[\Gamma]_\delta^\Sigma$ .

We instantiate  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : s_1^\dagger, \Gamma \vDash^\dagger \mathbf{m}_1 : s^\dagger$  with  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,

$(\mathbf{W}', \gamma[x_1 \mapsto \mathbf{m}'']) \in \mathcal{G}^\dagger[\Gamma_1^\dagger, \mathbf{x}_1 : s_1^\dagger]_\delta^\Sigma$ ,  $(\mathbf{W}', \gamma) \in \mathcal{G}^\dagger[\Gamma]_\delta^\Sigma$  and conclude:

$(\mathbf{W}', \delta(\gamma[x_1 \mapsto \mathbf{m}''])(\gamma(\mathbf{m}_1))) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

Note that this is the same as  $(\mathbf{W}', \mathbf{m}'_1[\mathbf{m}''/x_1]) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

**Case**  $i = 2$

From  $(\mathbf{W}', \text{inj}_i \mathbf{m}'') \in \mathcal{V}^\dagger[s_1^\dagger + t_2]_\delta^\Sigma$  we know  $(\mathbf{W}', \mathbf{m}'') \in \mathcal{E}^\dagger[t_2]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, \mathbf{x}_2 : t_2 \vdash \mathbf{m}_2 : s^\dagger$ , we know

$\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, \mathbf{x}_2 : t_2 \vDash^\dagger \mathbf{m}_2 : s^\dagger$ .

We instantiate this with  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,  $(\mathbf{W}', \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]_\delta^\Sigma$ ,

$(\mathbf{W}', \gamma[x_2 \mapsto \mathbf{m}'']) \in \mathcal{G}^\dagger[\Gamma, \mathbf{x}_2 : t_2]_\delta^\Sigma$  and conclude:

$(\mathbf{W}', \delta(\gamma(\gamma[x_2 \mapsto \mathbf{m}''])(\mathbf{m}_2))) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

Note that this is exactly  $(\mathbf{W}', \mathbf{m}'_2[\mathbf{m}''/x_2]) \in \mathcal{E}^\dagger[s^\dagger]_\delta^\Sigma$ .

**Case**  $\exists s_2^\dagger. t_2 = s_2^\dagger$ , but  $\nexists s_1^\dagger. t_1 = s_1^\dagger$ .

The proof of this case is symmetric to the previous case. The primary difference is the cases for  $i = 1$  and  $i = 2$  are switched.

**Case**  $\nexists s_1^\dagger, s_2^\dagger. t_1 = s_1^\dagger$  or  $t_2 = s_2^\dagger$

This case is analogous to the previous cases. We repeat it since both types are non-

translation types and it is not obviously symmetric.

We must show  $(\mathbf{W}, \text{case } m' \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m : t_1 + t_2$ , we know  $(\mathbf{W}, m') \in \mathcal{E}^\uparrow[[t_1 + t_2]]_\delta^\Sigma$ .

By  $(\mathbf{W}, m') \in \mathcal{E}^\uparrow[[t_1 + t_2]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}, \text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2) \in \mathcal{K}^\uparrow[[t_1 + t_2, s^\dagger]]_\delta^\Sigma$

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \text{inj}_i m'') \in \mathcal{V}^\uparrow[[t_1 + t_2]]_\delta^\Sigma$ , we must show

$(\mathbf{W}', \text{case inj}_i m'' \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2) \in \mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By FD-Val and the definition of  $\mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}', m'_i[m''/x_i]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

Recall  $\mathbf{W}' \supseteq \mathbf{W}$ , so by Lemma 5.26 (Monotonicity of  $\mathcal{E}^\uparrow[[t]]_\delta^\Sigma$ ), and the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, x_i : t_i \vdash m_i : t$ , we know  $(\mathbf{W}', \gamma[x_i \mapsto m''] (m'_i)) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

Note that this is exactly  $(\mathbf{W}', m'_i[m''/x_i]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

**Case**  $\not\exists s. t = s^\dagger$

We must show  $(\mathbf{W}, \text{case } m' \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2) \in \mathcal{E}^\uparrow[[t]]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, E^\#) \in \mathcal{K}^\uparrow[[t, s^\dagger]]_\delta^\Sigma$ , we must show

$(\mathbf{W}_1, E^\#[\text{case } m' \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2]) \in \mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

**Case**  $\exists s_1, s_1.t_1 + t_2 = s_1^\dagger + s_2^\dagger$

Note that in this case  $E^\#[\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2] \neq E_1^\#$ , since

$\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2 : s_1^\dagger + s_2^\dagger \Rightarrow t$ . Therefore FD-Val cannot apply.

By FD-Stuck, it suffices to show  $(\mathbf{W}_1, \text{case } m' \text{ of inj}_1 x_1. E^\#[m'_1] \parallel \text{inj}_2 x_2. E^\#[m'_2]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ . (We ignore alpha-renaming w.l.o.g.)

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m : t_1 + t_2$ , we know  $(\mathbf{W}_1, m') \in \mathcal{E}^\uparrow[[s_1^\dagger + s_2^\dagger]]_\delta^\Sigma$ .

Let  $\mathbf{W}_2 = (\mathbf{W}_1, x_i : t_i)$

By Lemma 5.28 (Monotonicity of  $\mathcal{G}^\uparrow[[\Gamma]]_\delta^\Sigma$ ), we know  $(\mathbf{W}_2, \gamma) \in \mathcal{G}^\uparrow[[\Gamma_1^\dagger]]_\delta^\Sigma$  and  $(\mathbf{W}_2, \gamma) \in \mathcal{G}^\uparrow[[\Gamma]]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, x_i : t_i \vdash m_i : t$ , we know  $(\mathbf{W}_2, m'_i) \in \mathcal{E}^\uparrow[[t]]_\delta^\Sigma$ .

By Lemma 5.27 (Monotonicity of  $\mathcal{K}^\uparrow[[t, s^\dagger]]_\delta^\Sigma$ ), we know  $(\mathbf{W}_2, E^\#) \in \mathcal{K}^\uparrow[[t, s^\dagger]]_\delta^\Sigma$ .

Instantiating  $(\mathbf{W}_2, m'_i) \in \mathcal{E}^\uparrow[[t]]_\delta^\Sigma$  with  $(\mathbf{W}_2, E^\#)$ , we know  $(\mathbf{W}_2, E^\#[m'_i]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By FD-Subterm and FD-Case,  $(\mathbf{W}_1, \text{case } m' \text{ of inj}_1 x_1. E^\#[m'_1] \parallel \text{inj}_2 x_2. E^\#[m'_2]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

**Case**  $\exists s_1^\dagger.t_1 = s_1^\dagger$ , but  $\not\exists s_2^\dagger.t_2 = s_2^\dagger$ .

We must show  $(\mathbf{W}_1, E^\#[\text{case } m' \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2]) \in \mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m' : t_1 + t_2$ , we know  $(\mathbf{W}_1, m') \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ .

By  $(\mathbf{W}_1, m') \in \mathcal{E}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, E^\#[\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2]) \in \mathcal{K}^\uparrow[[s_1^\dagger + t_2, s^\dagger]]_\delta^\Sigma$ .

Note that  $E^\#[\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2] = E_1^\#$  since  $\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 x_1. m'_1 \parallel \text{inj}_2 x_2. m'_2 : s_1^\dagger + t_2 \Rightarrow t$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \text{inj}_i m'') \in \mathcal{V}^\uparrow[[s_1^\dagger + t_2]]_\delta^\Sigma$ , we must show



$(\mathbf{W}_2, \mathbf{E}^\#[\text{case inj}_i \mathbf{m}'' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

By FD-Val, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_i[\mathbf{m}''/\mathbf{x}_i]]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

**Case  $i = 1$**

Note that  $(\mathbf{W}_2, \mathbf{m}'') \in \mathcal{E}^\uparrow[\![\mathbf{s}_1^\dagger]\!]_\delta^\Sigma$  and  $\mathbf{t}_1 = \mathbf{s}_1^\dagger$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : \mathbf{s}_1^\dagger, \Gamma \vdash \mathbf{m}_1 : \mathbf{t}$ , we know

$\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : \mathbf{s}_1^\dagger, \Gamma \vDash^\uparrow \mathbf{m}_1 : \mathbf{t}$ .

Recall that  $\mathbf{W}_2 \supseteq \mathbf{W}$ . So by Lemma 5.28 (Monotonicity of  $\mathcal{G}^\uparrow[\![\Gamma]\!]_\delta^\Sigma$ ), we know

$(\mathbf{W}_2, \gamma) \in \mathcal{G}^\uparrow[\![\Gamma]\!]_\delta^\Sigma, (\mathbf{W}_2, \gamma[\mathbf{x}_1 \mapsto \mathbf{m}'']) \in \mathcal{G}^\uparrow[\![\Gamma_1^\dagger, \mathbf{x}_1 : \mathbf{s}_1^\dagger]\!]_\delta^\Sigma$ .

We instantiate  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_1 : \mathbf{s}_1^\dagger, \Gamma \vDash^\uparrow \mathbf{m}_1 : \mathbf{t}$  with  $\delta \in \mathcal{D}^\uparrow[\![\Delta]\!]^\Sigma$ ,

$(\mathbf{W}_2, \gamma) \in \mathcal{G}^\uparrow[\![\Gamma]\!]_\delta^\Sigma, (\mathbf{W}_2, \gamma[\mathbf{x}_1 \mapsto \mathbf{m}'']) \in \mathcal{G}^\uparrow[\![\Gamma_1^\dagger, \mathbf{x}_1 : \mathbf{s}_1^\dagger]\!]_\delta^\Sigma$ .

Therefore we know  $(\mathbf{W}_2, \gamma[\mathbf{x}_1 \mapsto \mathbf{m}''])(\mathbf{m}'_1) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ . Note that this is exactly

$(\mathbf{W}_2, \mathbf{m}'_1[\mathbf{m}''/\mathbf{x}_1]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ .

Instantiating  $(\mathbf{W}_2, \mathbf{m}'_1[\mathbf{m}''/\mathbf{x}_1]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$  with  $\mathbf{E}^\#$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_1[\mathbf{m}''/\mathbf{x}_1]]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

**Case  $i = 2$**

Note that  $(\mathbf{W}_2, \mathbf{m}'') \in \mathcal{E}^\uparrow[\![\mathbf{t}_2]\!]_\delta^\Sigma$

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, \mathbf{x}_2 : \mathbf{t}_2 \vdash \mathbf{m}_2 : \mathbf{t}$ , we know  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_2 : \mathbf{t}_2 \vDash^\uparrow \mathbf{m}_2 : \mathbf{t}$ .

We instantiate this with  $\delta \in \mathcal{D}^\uparrow[\![\Delta]\!]^\Sigma, (\mathbf{W}_2, \gamma[\mathbf{x}_2 \mapsto \mathbf{m}'']) \in \mathcal{G}^\uparrow[\![\Gamma, \mathbf{x}_2 : \mathbf{t}_2]\!]_\delta^\Sigma, (\mathbf{W}_2, \gamma) \in \mathcal{G}^\uparrow[\![\Gamma_1^\dagger]\!]_\delta^\Sigma$ .

Therefore we know  $(\mathbf{W}_2, \gamma[\mathbf{x}_2 \mapsto \mathbf{m}''])(\mathbf{m}'_2) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ . Note that this is exactly

$(\mathbf{W}_2, \mathbf{m}'_2[\mathbf{m}''/\mathbf{x}_2]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ .

Instantiating  $(\mathbf{W}_2, \mathbf{m}'_2[\mathbf{m}''/\mathbf{x}_2]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$  with  $\mathbf{E}^\#$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_2[\mathbf{m}''/\mathbf{x}_2]]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

**Case  $\exists \mathbf{s}_2^\dagger. \mathbf{t}_2 = \mathbf{s}_2^\dagger$ , but  $\nexists \mathbf{s}_1^\dagger. \mathbf{t}_1 = \mathbf{s}_1^\dagger$ .**

The proof of this case is symmetric to the previous case. The primary difference is the cases for  $i = 1$  and  $i = 2$  are switched.

**Case  $\nexists \mathbf{s}_1^\dagger, \mathbf{s}_2^\dagger. \mathbf{t}_1 = \mathbf{s}_1^\dagger$  or  $\mathbf{t}_2 = \mathbf{s}_2^\dagger$**

This case is analogous to the previous cases. We repeat it since both types are non-translation types and it is not obviously symmetric.

We must show  $(\mathbf{W}_1, \mathbf{E}^\#[\text{case } \mathbf{m}' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$ , we know  $(\mathbf{W}_1, \mathbf{m}') \in$

$\mathcal{E}^\uparrow[\![\mathbf{t}_1 + \mathbf{t}_2]\!]_\delta^\Sigma$ .

By  $(\mathbf{W}_1, \mathbf{m}') \in \mathcal{E}^\uparrow[\![\mathbf{t}_1 + \mathbf{t}_2]\!]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\#[\text{case } [\cdot]_{\mathbf{T}} \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1, (\mathbf{W}_2, \text{inj}_i \mathbf{m}'') \in \mathcal{V}^\uparrow[\![\mathbf{t}_1 + \mathbf{t}_2]\!]_\delta^\Sigma$ , we must show

$(\mathbf{W}_2, \mathbf{E}^\#[\text{case inj}_i \mathbf{m}'' \text{ of inj}_1 \mathbf{x}_1. \mathbf{m}'_1 \parallel \text{inj}_2 \mathbf{x}_2. \mathbf{m}'_2]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

By FD-Val, it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}_i[\mathbf{m}''/\mathbf{x}_i]]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x}_i : \mathbf{t}_i, \Gamma \vdash \mathbf{m}_i : \mathbf{t}$ , we know  $(\mathbf{W}_2, \gamma[\mathbf{x}_i \mapsto \mathbf{m}'']) (\mathbf{m}'_i) \in$

$\mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ . Note that this is exactly  $(\mathbf{W}_2, \mathbf{m}'_i[\mathbf{m}''/\mathbf{x}_i]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$ .

Instantiating  $(\mathbf{W}_2, \mathbf{m}'_i[\mathbf{m}''/\mathbf{x}_i]) \in \mathcal{E}^\uparrow[\![\mathbf{t}]\!]_\delta^\Sigma$  with  $\mathbf{E}^\#$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}'_i[\mathbf{m}''/\mathbf{x}_i]]) \in \mathcal{E}^\uparrow[\![\mathbf{s}^\dagger]\!]_\delta^\Sigma$ .

Case FT-Fun

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x} : \mathbf{t}_1, \Gamma \vdash \mathbf{m} : \mathbf{t}_2 \quad \mathbf{D}_\ell, \Delta \vdash \mathbf{t}_1 :: *}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \lambda \mathbf{x} : \mathbf{t}_1. \mathbf{m} : \mathbf{t}_1 \rightarrow \mathbf{t}_2}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\dagger \lambda \mathbf{x} : \mathbf{t}_1. \mathbf{m} : \mathbf{t}_1 \rightarrow \mathbf{t}_2$ .

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma]^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\gamma(\lambda \mathbf{x} : \mathbf{t}_1. \mathbf{m})))) \in \mathcal{E}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{t}_2]_\delta^\Sigma.$$

For brevity, let  $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{m})))$ , and  $\mathbf{t}'_1 = \delta(\mathbf{t}_1)$ .

Case  $\exists s_1. \mathbf{t}_1 = s_1$ , and  $\exists s_2. \mathbf{t}_2 = s_2$

We must show  $(\mathbf{W}, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{s}_2^\dagger]_\delta^\Sigma$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x} : \mathbf{t}_1, \Gamma \vdash \mathbf{m} : \mathbf{t}_2$  we know:  $((\mathbf{W}, \mathbf{x} : \mathbf{s}_1^\dagger), \mathbf{m}) \in \mathcal{E}^\dagger[\mathbf{s}_2^\dagger]_\delta^\Sigma$ .

By FD-Fun, we know  $(\mathbf{W}, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{s}_2^\dagger]_\delta^\Sigma$ .

Case  $\exists s_1. \mathbf{t}_1 = s_1$ , but  $\nexists s_2. \mathbf{t}_2 = s_2$

We must show  $(\mathbf{W}, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{t}_2]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{t}_2, \mathbf{s}^\dagger]_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\#[\lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}']) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]_\delta^\Sigma$ .

By the definition of  $\mathcal{K}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{t}_2, \mathbf{s}^\dagger]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{V}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{t}_2]_\delta^\Sigma$ .

By the definition of  $\mathcal{V}^\dagger[\mathbf{s}_1^\dagger \rightarrow \mathbf{t}_2]_\delta^\Sigma$ , it suffices to show, for arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}_1) \in \mathcal{E}^\dagger[\mathbf{s}_1^\dagger]_\delta^\Sigma$ , that  $(\mathbf{W}_2, \mathbf{m}'[\mathbf{x}/\mathbf{m}_1]) \in \mathcal{E}^\dagger[\mathbf{t}_2]_\delta^\Sigma$ , which follows by the IH applied to

$\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \mathbf{x} : \mathbf{s}_1^\dagger, \Gamma \vdash \mathbf{m} : \mathbf{t}_2$ , with  $(\mathbf{W}_2, \gamma[\mathbf{x} \mapsto \mathbf{m}_1]) \in \mathcal{G}^\dagger[\Gamma_1^\dagger, \mathbf{x} : \mathbf{s}_1^\dagger]_\delta^\Sigma$ .

Case  $\exists s_2. \mathbf{t}_2 = s_2$ , but  $\nexists s_1. \mathbf{t}_1 = s_1$

We must show  $(\mathbf{W}, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{E}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{s}_2^\dagger]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{s}_2^\dagger, \mathbf{s}^\dagger]_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\#[\lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}']) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]_\delta^\Sigma$ .

By the definition of  $\mathcal{K}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{s}_2^\dagger, \mathbf{s}^\dagger]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \lambda \mathbf{x} : \mathbf{t}'_1. \mathbf{m}') \in \mathcal{V}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{s}_2^\dagger]_\delta^\Sigma$ .

By the definition of  $\mathcal{V}^\dagger[\mathbf{t}_1 \rightarrow \mathbf{s}_2^\dagger]_\delta^\Sigma$ , it suffices to show, for arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{m}_1) \in \mathcal{E}^\dagger[\mathbf{t}_1]_\delta^\Sigma$ , that  $(\mathbf{W}_2, \mathbf{m}'[\mathbf{x}/\mathbf{m}_1]) \in \mathcal{E}^\dagger[\mathbf{s}_2^\dagger]_\delta^\Sigma$ , which follows by the IH applied to

$\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma, \mathbf{x} : \mathbf{s}_1^\dagger \vdash \mathbf{m} : \mathbf{t}_2$ , with  $(\mathbf{W}_2, \gamma[\mathbf{x} \mapsto \mathbf{m}_1]) \in \mathcal{G}^\dagger[\Gamma, \mathbf{x} : \mathbf{t}_1]_\delta^\Sigma$ .

Case  $\nexists s_1. \mathbf{t}_1 = s_1$ , and  $\nexists s_2. \mathbf{t}_2 = s_2$

This case is exactly analogous to the previous cases.

Case FT-App

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_1 : \mathbf{t}_1 \rightarrow \mathbf{t}_2 \quad \mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_2 : \mathbf{t}_1}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_1 \mathbf{m}_2 : \mathbf{t}_2}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\dagger \mathbf{m}_1 \mathbf{m}_2 : \mathbf{t}_2$ .

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\dagger[\Delta]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma_1^\dagger]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\dagger[\Gamma]^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\gamma(\mathbf{m}_1 \mathbf{m}_2)))) \in \mathcal{E}^\dagger[\mathbf{t}_2]_\delta^\Sigma.$$

For brevity, let  $\mathbf{m}'_1 = \delta(\gamma(\gamma(\mathbf{m}_1)))$ , and  $\mathbf{m}'_2 = \delta(\gamma(\gamma(\mathbf{m}_2)))$ .

Case  $\exists s_2. t_2 = s_2^\dagger$

Case  $\exists s_1. t_1 = s_1^\dagger$ .

We must show

$$(\mathbf{W}, \mathbf{m}'_1 \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[s_2^\dagger]]_\delta^\Sigma.$$

By FD-Subterm and FD-App, it suffices to show  $(\mathbf{W}, \mathbf{m}'_1) \in \mathcal{E}^\uparrow[[s_1^\dagger \rightarrow s_2^\dagger]]_\delta^\Sigma$  and  $(\mathbf{W}, \mathbf{m}'_2) \in$

$$\mathcal{E}^\uparrow[[s_1^\dagger]]_\delta^\Sigma, \text{ which follow by the IH applied to } \mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_1 : t_1 \rightarrow t_2 \text{ and } \mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_2 : t_1$$

Case  $\nexists s_1. t_1 = s_1^\dagger$ .

We must show

$$(\mathbf{W}, \mathbf{m}'_1 \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[s_2^\dagger]]_\delta^\Sigma.$$

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_1 : t_1 \rightarrow s_2^\dagger$ , we know  $(\mathbf{W}, \mathbf{m}'_1) \in$

$$\mathcal{E}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma.$$

By definition of  $\mathcal{E}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}, [\cdot]_{\mathbf{T}} \mathbf{m}'_2) \in \mathcal{K}^\uparrow[[t_1 \rightarrow s_2^\dagger, s^\dagger]]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \lambda x:t'_1. \mathbf{m}) \in \mathcal{V}^\uparrow[[s_1^\dagger \rightarrow t_2]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}', (\lambda x:t'_1. \mathbf{m}) \mathbf{m}'_2) \in$

$$\mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma.$$

Note that  $(\lambda x:t'_1. \mathbf{m}) \mathbf{m}'_2 = \mathbf{E}^\#[\mathbf{u}]$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_2 : t_1$ , we know  $(\mathbf{W}', \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[t_1]]_\delta^\Sigma$ .

By FD-Val and the definition of  $\mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}', \mathbf{m}[\mathbf{m}'_2/x]) \in \mathcal{E}^\uparrow[[s_2^\dagger]]_\delta^\Sigma$ ,

which follows by instantiating  $(\mathbf{W}', \lambda x:t'_1. \mathbf{m}) \in \mathcal{V}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma$  with  $(\mathbf{W}', \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[t_1]]_\delta^\Sigma$ .

Case  $\nexists s_2. t_2 = s_2^\dagger$

We must show

$$(\mathbf{W}, \mathbf{m}'_1 \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[t_2]]_\delta^\Sigma.$$

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow[[t_2, s^\dagger]]_\delta^\Sigma$ , it suffice to show  $(\mathbf{W}_1, \mathbf{E}^\#[\mathbf{m}'_1 \mathbf{m}'_2]) \in$

$$\mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma.$$

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_1 : t_1 \rightarrow s_2^\dagger$ , we know  $(\mathbf{W}_1, \mathbf{m}'_1) \in$

$$\mathcal{E}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma.$$

By definition of  $\mathcal{E}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \mathbf{E}^\#[[\cdot]_{\mathbf{T}} \mathbf{m}'_2]) \in \mathcal{K}^\uparrow[[t_1 \rightarrow s_2^\dagger, s^\dagger]]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \lambda x:t'_1. \mathbf{m}) \in \mathcal{V}^\uparrow[[s_1^\dagger \rightarrow t_2]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[(\lambda x:t'_1. \mathbf{m}) \mathbf{m}'_2]) \in$

$$\mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma.$$

Note that  $\mathbf{E}^\#[(\lambda x:t'_1. \mathbf{m}) \mathbf{m}'_2] = \mathbf{E}_1^\#[\mathbf{u}]$ .

By the IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}_2 : t_1$ , we know  $(\mathbf{W}_2, \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[t_1]]_\delta^\Sigma$ .

By FD-Val and the definition of  $\mathcal{E}^\uparrow[[s^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_2, \mathbf{E}^\#[\mathbf{m}[\mathbf{m}'_2/x]]) \in \mathcal{O}^\uparrow[[s^\dagger]]_\delta^\Sigma$ .

By Lemma 5.27 applied to  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\uparrow[[t_2, s^\dagger]]_\delta^\Sigma$ , we conclude  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\uparrow[[t_2, s^\dagger]]_\delta^\Sigma$ .

By  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\uparrow[[t_2, s^\dagger]]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_2, \mathbf{m}[\mathbf{m}'_2/x]) \in \mathcal{E}^\uparrow[[t_2]]_\delta^\Sigma$ , which follows by instantiating  $(\mathbf{W}_2, \lambda x:t'_1. \mathbf{m}) \in \mathcal{V}^\uparrow[[t_1 \rightarrow s_2^\dagger]]_\delta^\Sigma$  with  $(\mathbf{W}_2, \mathbf{m}'_2) \in \mathcal{E}^\uparrow[[t_1]]_\delta^\Sigma$ .

Case FT-Abs

$$\frac{\mathbf{D}_\ell, \alpha :: \kappa, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : t}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \Lambda \alpha :: \kappa. \mathbf{m} : \forall \alpha :: \kappa. t}$$

We must show  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vDash^\uparrow \Lambda \alpha :: \kappa. \mathbf{m} : \forall \alpha :: \kappa. t$

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \gamma$ . such that  $\delta \in \mathcal{D}^\uparrow[[\Delta]]_\delta^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[[\Gamma_1^\dagger]]_\delta^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow[[\Gamma]]_\delta^\Sigma$ , we must show

$(\mathbf{W}, \delta(\gamma(\gamma(\Lambda\alpha::\kappa.\mathbf{m})))) \in \mathcal{E}^\dagger[\forall\alpha::\kappa. \mathbf{t}]_\delta^\Sigma$ .

For brevity, let  $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{m})))$ .

**Case**  $\exists \mathbf{T}_\ell s. \forall \alpha::\kappa. \mathbf{t} = (\mathbf{T}_\ell s)^\dagger$ .

We must show  $(\mathbf{W}, \Lambda\alpha::\kappa.\mathbf{m}) \in \mathcal{E}^\dagger[(\mathbf{T}_\ell s)^\dagger]_\delta^\Sigma$ .

By FD-Return, it suffices to show  $((\mathbf{W}_k, \mathbf{k} : \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell s)^\dagger; \mathbf{W}_\Gamma), \mathbf{m}'[(\mathbf{T}_\ell s)^\dagger/\alpha] \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{E}^\dagger[(\mathbf{T}_\ell s)^\dagger]_\delta^\Sigma$

Let  $\mathbf{m}'[(\mathbf{T}_\ell s)^\dagger/\alpha] = \mathbf{m}''$ .

Let  $\delta' = \delta[\alpha \mapsto ((\mathbf{T}_\ell s)^\dagger, \mathcal{E}^\dagger[(\mathbf{T}_\ell s)^\dagger]_\delta^\Sigma)]$ .

By the IH applied to  $\mathbf{D}_\ell, \alpha :: *, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m} : ((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha$ , we conclude  $(\mathbf{W}, \mathbf{m}'') \in \mathcal{E}^\dagger[((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha]_\delta^\Sigma \delta'$ .

By Lemma 5.26 (Monotonicity of  $\mathcal{E}^\dagger[\mathbf{t}]_\delta^\Sigma$ ), we know

$((\mathbf{W}_k, \mathbf{k} : \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell s)^\dagger; \mathbf{W}_\Gamma), \mathbf{m}'') \in \mathcal{E}^\dagger[((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha]_\delta^\Sigma \delta'$ .

Therefore, it suffices to show

$((\mathbf{W}_k, \mathbf{k} : \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell s)^\dagger; \mathbf{W}_\Gamma), [\cdot]_{\mathbf{T}} \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{K}^\dagger[((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha, \delta']_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq (\mathbf{W}_k, \mathbf{k} : \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell s)^\dagger; \mathbf{W}_\Gamma)$ ,

$(\mathbf{W}_1, \lambda\mathbf{x}:(\dots).\mathbf{m}_1) \in \mathcal{V}^\dagger[((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha]_\delta^\Sigma \delta'$ ,

show  $(\mathbf{W}_1, (\lambda\mathbf{x}:(\dots).\mathbf{m}_1) \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{E}^\dagger[\alpha]_\delta^\Sigma \delta'$ .

By definition of  $\mathcal{E}^\dagger[\alpha]_\delta^\Sigma$ , it suffices to show

$(\mathbf{W}_1, (\lambda\mathbf{x}:(\dots).\mathbf{m}_1) \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{E}^\dagger[(\mathbf{T}_\ell s)^\dagger]_\delta^\Sigma$ .

By FD-Val, it suffices to show  $(\mathbf{W}_1, \mathbf{m}_1[\langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle/\mathbf{x}]) \in \mathcal{E}^\dagger[(\mathbf{T}_\ell s)^\dagger]_\delta^\Sigma$ .

From  $(\mathbf{W}_1, \lambda\mathbf{x}:(\dots).\mathbf{m}) \in \mathcal{V}^\dagger[((\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)) \rightarrow \alpha]_\delta^\Sigma \delta'$  and definition of  $\mathcal{E}^\dagger[\alpha]_\delta^\Sigma$ , it suffices to show

$(\mathbf{W}_1, \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{E}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)]_\delta^\Sigma \delta'$ .

For arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha), \delta']_\delta^\Sigma$ , we must show

$(\mathbf{W}_2, \mathbf{E}^\#[\langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle]) \in \mathcal{O}^\dagger[\delta']_\delta^\Sigma$ .

By assumption that  $(\mathbf{W}_2, \mathbf{E}^\#) \in \mathcal{K}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha), \delta']_\delta^\Sigma$ , it suffices to show

$(\mathbf{W}_2, \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{V}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)]_\delta^\Sigma \delta'$ .

The proof that  $(\mathbf{W}_2, \mathbf{k}) \in \mathcal{E}^\dagger[\mathbf{s}^\dagger \rightarrow \alpha]_\delta^\Sigma \delta'$  is similar to the the FT-Var case for  $\mathbf{x} : \mathbf{s}^\dagger \rightarrow (\mathbf{T}_\ell s)^\dagger$ .

Recall that the type of  $\hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s]$  is  $(\hat{\alpha}_\leq \hat{\alpha}_\ell (\mathbf{T}_\ell s)^\dagger)$ , which cannot appear in evaluation position. Therefore,  $(\mathbf{W}_2, \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s]) \in \mathcal{E}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha)]_\delta^\Sigma \delta'$  is vacuously true.

Therefore,  $(\mathbf{W}_2, \langle \hat{\text{pf}}[\ell \leq \mathbf{T}_\ell s], \mathbf{k} \rangle) \in \mathcal{V}^\dagger[(\hat{\alpha}_\leq \hat{\alpha}_\ell \alpha) \times (\mathbf{s}^\dagger \rightarrow \alpha)]_\delta^\Sigma \delta'$ , completing this case of the proof.

**Case**  $\nexists \mathbf{T}_\ell s. \forall \alpha::\kappa. \mathbf{t} = \mathbf{T}_\ell s^\dagger$ .

We must show  $(\mathbf{W}, \Lambda\alpha::\kappa.\mathbf{m}') \in \mathcal{E}^\dagger[\forall\alpha::\kappa. \mathbf{t}]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}_1 \supseteq \mathbf{W}$ ,  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\dagger[\forall\alpha::\kappa. \mathbf{t}, \mathbf{s}^\dagger]_\delta^\Sigma$ , we must show  $(\mathbf{W}_1, \mathbf{E}^\#[\Lambda\alpha::\kappa.\mathbf{m}']) \in \mathcal{O}^\dagger[\mathbf{s}^\dagger]_\delta^\Sigma$ .

By definition of  $(\mathbf{W}_1, \mathbf{E}^\#) \in \mathcal{K}^\dagger[\forall\alpha::\kappa. \mathbf{t}, \mathbf{s}^\dagger]_\delta^\Sigma$ , it suffices to show  $(\mathbf{W}_1, \Lambda\alpha::\kappa.\mathbf{m}') \in \mathcal{V}^\dagger[\forall\alpha::\kappa. \mathbf{t}]_\delta^\Sigma$ .

To show  $(\mathbf{W}_1, \Lambda\alpha::\kappa.\mathbf{m}') \in \mathcal{V}^\dagger[\forall\alpha::\kappa. \mathbf{t}]_\delta^\Sigma$ , it suffices to show, for arbitrary  $\mathbf{W}_2 \supseteq \mathbf{W}_1$ ,

$(\mathbf{t}_1, \mathbf{R}) \in \text{Rel}^\dagger_{\kappa}^\Sigma$ ,  $(\mathbf{W}_2, \mathbf{m}'[\mathbf{t}_1/\alpha]) \in \mathcal{E}^\dagger[\mathbf{t}[\mathbf{t}_1/\alpha]]_\delta^\Sigma \delta[\alpha \mapsto (\mathbf{t}', \mathbf{R})]$ , which follows by IH applied to  $\mathbf{D}_\ell, \alpha :: \kappa, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash \mathbf{m}' : \mathbf{t}$ .

**Case** FT-Inst We know

$$\frac{D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash m : \forall \alpha :: \kappa. t_1 \quad D_\ell, \Delta \vdash t_2 :: \kappa}{D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash m[t_2] : t_1[t_2/\alpha]}$$

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \Upsilon$ . such that  $\delta \in \mathcal{D}^\uparrow[\Delta]^\Sigma$ ,  $(W, \gamma) \in \mathcal{G}^\uparrow[\Gamma_1^\dagger]^\Sigma$ ,  $(W, \Upsilon) \in \mathcal{G}^\uparrow[\Gamma]^\Sigma$ , we must show

$$(W, \delta(\gamma(\Upsilon(m[t_2]))) \in \mathcal{E}^\uparrow[t_1[t_2/\alpha]]^\Sigma.$$

For brevity, let  $m' = \delta(\gamma(\Upsilon(m)))$ .

**Case**  $\exists s. t_1[t_2/\alpha] = s^\dagger$ .

We must show  $(W, m[t_2]) \in \mathcal{E}^\uparrow[s^\dagger]^\Sigma$ .

Note then that  $\exists s. \forall \alpha :: \kappa. t = (T_\ell s)^\dagger$ .

By the IH applied to  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash m : \forall \alpha :: \kappa. t$ , we know  $(W, m') \in \mathcal{E}^\uparrow[\forall \alpha :: \kappa. t]^\Sigma$ .

By  $(W, m') \in \mathcal{E}^\uparrow[\forall \alpha :: \kappa. t]^\Sigma$ , it suffices to show  $(W, [\cdot]_T[t_2]) \in \mathcal{K}^\uparrow[\forall \alpha :: \kappa. t, s^\dagger]^\Sigma$ .

For arbitrary  $W' \supseteq W$ ,  $(W', \Lambda \alpha :: \kappa. m'') \in \mathcal{V}^\uparrow[\forall \alpha :: \kappa. t]^\Sigma$ , we must show  $(W', (\Lambda \alpha :: \kappa. m'')[t_2]) \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ .

Note that  $[\cdot]_T[t_2] = E^\#$ . Therefore, by FD-Val, it suffices to show  $(W', m''[t_2/\alpha]) \in \mathcal{E}^\uparrow[s^\dagger]^\Sigma$ .

By Lemma 5.31 (back-translation type interpretation is well-formed) applied to  $\Delta \vdash t_2 :: \kappa$  we know  $(t_2, \mathcal{T}^\uparrow[t_2 :: \kappa_1 \rightarrow \kappa_2]^\Sigma) \in Rel^\uparrow \frac{\Sigma}{\kappa}$ .

Instantiating  $(W', \Lambda \alpha :: \kappa. m'') \in \mathcal{V}^\uparrow[\forall \alpha :: \kappa. t]^\Sigma$  with  $(t_2, \mathcal{T}^\uparrow[t_2 :: \kappa_1 \rightarrow \kappa_2]^\Sigma)$ , we conclude  $(W', m''[t_2/\alpha]) \in \mathcal{E}^\uparrow[t_1]^\Sigma \delta [\alpha \mapsto (t_2, \mathcal{T}^\uparrow[t_2 :: \kappa]^\Sigma)]$ .

By Lemma 5.32 (Compositionality), we conclude  $(W', m''[t_2/\alpha]) \in \mathcal{E}^\uparrow[s^\dagger]^\Sigma$ .

**Case**  $\exists s. t_1[t_2/\alpha] = s^\dagger$ .

We must show  $(W, m'[t_2]) \in \mathcal{E}^\uparrow[t_1[t_2/\alpha]]^\Sigma$ .

For arbitrary  $W_1 \supseteq W$ ,  $(W_1, E^\#) \in \mathcal{K}^\uparrow[t, s^\dagger]^\Sigma$ , we must show  $(W_1, E^\#[m'[t_2]]) \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ .

**Case**  $\exists s. \forall \alpha :: *. t_1 = (T_\ell s)^\dagger$ .

Then  $E^\#$  must equal  $[\cdot]_T m''$  where  $\Sigma_D; \Sigma_G, G_k, \Gamma^\dagger \vdash m'' : (\hat{\alpha}_{\leq} \hat{\alpha}_\ell (T_\ell s)^\dagger) \times (s^\dagger \rightarrow (T_\ell s)^\dagger)$ .

By FD-Bind, it suffices to show  $(W_1, m') \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ , which follows by IH applied to

$D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash m : \forall \alpha :: *. t_1$ , and  $(W_1, \text{prj}_2 m_p) \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ .

By  $(W_1, E^\#) \in \mathcal{K}^\uparrow[t, s^\dagger]^\Sigma$ , it suffices to show  $(W_1, \lambda x: (\hat{\alpha}_{\leq} \hat{\alpha}_\ell (T_\ell s)^\dagger) \times (s^\dagger \rightarrow (T_\ell s)^\dagger). \text{prj}_2 x) \in \mathcal{V}^\uparrow[(\dots)]^\Sigma$ .

For arbitrary  $W_2 \supseteq W_1$ ,  $(W_2, m_1) \in \mathcal{E}^\uparrow[(\hat{\alpha}_{\leq} \hat{\alpha}_\ell (T_\ell s)^\dagger) \times (s^\dagger \rightarrow (T_\ell s)^\dagger)]^\Sigma$ , we must show  $(W_2, \text{prj}_2 m_1) \in \mathcal{E}^\uparrow[s^\dagger \rightarrow (T_\ell s)^\dagger]^\Sigma$ .

It suffices to show  $(W_2, \text{prj}_2 [\cdot]_T) \in \mathcal{K}^\uparrow[s^\dagger \rightarrow (T_\ell s)^\dagger, s^\dagger]^\Sigma$ .

For arbitrary  $W' \supseteq W_2$ ,  $(W', m_2) \in \mathcal{V}^\uparrow[(\hat{\alpha}_{\leq} \hat{\alpha}_\ell (T_\ell s)^\dagger) \times (s^\dagger \rightarrow (T_\ell s)^\dagger)]^\Sigma$ , we

must show  $(W', \text{prj}_2 m_2) \in \mathcal{O}^\uparrow[s^\dagger]^\Sigma$ , which follows from FD-Val and assumption that

$(W', m_2) \in \mathcal{V}^\uparrow[(\hat{\alpha}_{\leq} \hat{\alpha}_\ell (T_\ell s)^\dagger) \times (s^\dagger \rightarrow (T_\ell s)^\dagger)]^\Sigma$ .

**Case**  $\exists s. \forall \alpha :: *. t_1 = (T_\ell s)^\dagger$ .

Note that  $E^\#[[\cdot]_T[t_2]] = E_1^\#$ .

By the IH applied to  $D_\ell, \Delta; G_\ell, G_{\leq}, G_k, \Gamma^\dagger, \Gamma \vdash m : \forall \alpha :: \kappa. t_1$ , we know  $(W, m') \in$

$\mathcal{E}^\uparrow[\forall \alpha :: \kappa. t_1]^\Sigma$ .

Therefore, it suffices to show  $(\mathbf{W}, \mathbf{E}^\# [[\cdot]_{\mathbf{T}} [t_2]]) \in \mathcal{K}^\uparrow [[\forall \alpha :: \kappa. t_1, s^\dagger]]_\delta^\Sigma$ .

For arbitrary  $\mathbf{W}' \supseteq \mathbf{W}$ ,  $(\mathbf{W}', \Lambda \alpha :: \kappa. m'') \in \mathcal{V}^\uparrow [[\forall \alpha :: \kappa. t_2]]_\delta^\Sigma$ , we must show  $(\mathbf{W}', \mathbf{E}^\# [(\Lambda \alpha :: \kappa. m'') [t_2]]) \in \mathcal{O}^\uparrow [[s^\dagger]]_\delta^\Sigma$ .

By FD-Val, it suffices to show  $(\mathbf{W}', \mathbf{E}^\# [m'' [t_2 / \alpha]]) \in \mathcal{O}^\uparrow [[s^\dagger]]_\delta^\Sigma$ , which follows by  $(\mathbf{W}', \Lambda \alpha :: \kappa. m'') \in \mathcal{V}^\uparrow [[\forall \alpha :: \kappa. t_1]]_\delta^\Sigma$ , Lemma 5.32 (back-translation compositionality), and Lemma 5.31 (back-translation type interpretation is well-formed) applied to  $\Delta \vdash t_2 :: \kappa$ .

**Case FT-Eqv**

We know

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m : t_1 \quad t_1 \equiv t_2 \quad \mathbf{D}_\ell, \Delta \vdash t_2 :: *}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m : t_2}$$

For arbitrary  $\Gamma_1^\dagger, \Gamma_2^\dagger, \delta, \gamma, \Upsilon$ . such that  $\delta \in \mathcal{D}^\uparrow [[\Delta]]^\Sigma$ ,  $(\mathbf{W}, \gamma) \in \mathcal{G}^\uparrow [[\Gamma_1^\dagger]]_\delta^\Sigma$ ,  $(\mathbf{W}, \Upsilon) \in \mathcal{G}^\uparrow [[\Gamma]]_\delta^\Sigma$ , we must show

$$(\mathbf{W}, \delta(\gamma(\Upsilon(m)))) \in \mathcal{E}^\uparrow [[t_2]]_\delta^\Sigma.$$

For brevity, let  $m' = \delta(\gamma(\Upsilon(m)))$ .

We must show  $(\mathbf{W}, m') \in \mathcal{E}^\uparrow [[t_2]]_\delta^\Sigma$ .

By IH applied to  $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_\leq, \mathbf{G}_k, \Gamma^\dagger, \Gamma \vdash m : t_1$ , we know  $(\mathbf{W}, m') \in \mathcal{E}^\uparrow [[t_1]]_\delta^\Sigma$ .

By Theorem 5.33 (Coherence for back-translation relation), we conclude  $(\mathbf{W}, m') \in \mathcal{E}^\uparrow [[t_2]]_\delta^\Sigma$ .  $\square$

**Theorem 5.35 (Back-translation exists)**

If  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m : s^\dagger$  then  $\exists e. \Sigma; \mathbf{G}_k, \Gamma^\dagger \vdash m : s^\dagger \uparrow e$ .

**Proof**

Follows from Lemma 5.34  $\square$

**Lemma 5.36 (Back-translation preserves semantics (generalized))**

Let  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_\leq \mapsto \hat{\alpha}_\leq\}$ ,  $\gamma_k = \{k : s^\dagger \rightarrow (T_\ell s)^\dagger \mapsto \eta_k^{\ell, s} \mid k : s^\dagger \rightarrow (T_\ell s)^\dagger \in \mathbf{G}_k\}$ .

If  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m : s^\dagger$  and  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m : s^\dagger \uparrow e$  then  $\Gamma \mid \Sigma \vdash e \simeq \gamma_k(m) : s \mid \delta$

**Proof**

By induction on the structure of  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m : s^\dagger \uparrow e$ .

**Case (FD-Unit)**  $\Sigma; \mathbf{G}_k, \Gamma^\dagger \vdash \langle \rangle : 1 \uparrow \langle \rangle$

For arbitrary  $\zeta$ ,  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+ [[\Gamma]]_\delta$ , must show  $(\langle \rangle, \langle \rangle) \in \mathcal{E}_\zeta^+ [[1]]_\delta$ .

Since  $\langle \rangle$  and  $\langle \rangle$  are values, it suffices to show  $(\langle \rangle, \langle \rangle) \in \mathcal{V}_\zeta^+ [[1]]_\delta$ , which follows by definition.

**Case (FD-Var)**  $\frac{(x : s^\dagger) \in \Gamma^\dagger}{\Sigma; \mathbf{G}_k, \Gamma^\dagger \vdash x : s^\dagger \uparrow x}$

For arbitrary  $\zeta$ ,  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+ [[\Gamma]]_\delta$ , must show  $(\gamma(x), \Upsilon(x)) \in \mathcal{E}_\zeta^+ [[s]]_\delta$ , which follows immediately from  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+ [[\Gamma]]_\delta$ .

**Case (FD-K)**  $\frac{k : (s^\dagger \rightarrow (T_\ell s)^\dagger) \in \mathbf{G}_k}{\Sigma; \mathbf{G}_k, \Gamma^\dagger \vdash k : s^\dagger \rightarrow T_\ell s^\dagger \uparrow \lambda x : s. \eta_\ell x}$

For arbitrary  $\zeta$ ,  $(\gamma, \Upsilon) \in \mathcal{G}_\zeta^+ [[\Gamma]]_\delta$ , must show  $(\lambda x : s. \eta_\ell x, \eta_k^{\ell, s}) \in \mathcal{E}_\zeta^+ [[s \rightarrow T_\ell s]]_\delta$ .

It suffices to show  $(\lambda x : s. \eta_\ell x, \eta_k^{\ell, s}) \in \mathcal{V}_\zeta^+[\![s \rightarrow T_\ell s]\!]_\delta$ .

For arbitrary  $(e', m') \in \mathcal{E}_\zeta^+[\![s]\!]_\delta$ , we must show  $(\eta_\ell e', \Lambda\beta::*. \lambda x: ((\alpha_{\leq} \alpha_\ell \beta) \times (s^\dagger \rightarrow \beta)). ((\text{prj}_2 x) m')) \in \mathcal{E}_\zeta^+[\![T_\ell s]\!]_\delta$ .

It suffices to show  $(\eta_\ell e', \Lambda\beta::*. \lambda x: ((\alpha_{\leq} \alpha_\ell \beta) \times (s^\dagger \rightarrow \beta)). ((\text{prj}_2 x) m')) \in \mathcal{V}_\zeta^+[\![T_\ell s]\!]_\delta$ .

Let  $\rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\rho^\Sigma$ .

By  $\mathcal{V}_\zeta^+[\![T_\ell s]\!]_\delta$ , it suffices to show  $\exists m''. \Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m'' : s^\dagger$  such that

- $((\lambda x: ((\alpha_{\leq} \alpha_\ell \ (\dagger T_\ell s)) \times s^\dagger \rightarrow (\dagger T_\ell s)). ((\text{prj}_2 x) m')) \langle \text{pf}[\![\ell \leq T_\ell s]\!], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} m'') \in \mathcal{E}[\![\langle T_\ell s \rangle^+]\!]_\rho^\Sigma$   
By evaluation we see that  $((\lambda x: ((\alpha_{\leq} \alpha_\ell \ (\dagger T_\ell s)) \times s^\dagger \rightarrow (\dagger T_\ell s)). ((\text{prj}_2 x) m')) \langle \text{pf}[\![\ell \leq T_\ell s]\!], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} m''$  is equivalent to  $(\eta_k^{\ell, s} m', \eta_k^{\ell, s} m'')$ .  
So, let  $m'' = m'$ . By Theorem 5.14, we know  $(\eta_k^{\ell, s} m', \eta_k^{\ell, s} m') \in \mathcal{E}[\![\langle T_\ell s \rangle^+]\!]_\rho^\Sigma$ .
- $(e', m'') \in \mathcal{E}_\zeta^+[\![s^+]\!]_\delta$   
Since  $m'' = m'$ , this hold by assumption.

$$\text{Case (FD-Sum)} \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash m : s_1^\dagger \uparrow e \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{inj}_i m : s_1^\dagger + s_2^\dagger \uparrow \text{inj}_i e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{inj}_i m : s_1^\dagger + s_2^\dagger \uparrow \text{inj}_i e}$$

For arbitrary  $\zeta, (\gamma, \Upsilon) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_\delta$ , we must show  $(\text{inj}_i \gamma(e), \text{inj}_i \Upsilon(\Upsilon_k(m))) \in \mathcal{E}_\zeta^+[\![s_1 + s_2]\!]_\delta$ .

Since  $\text{inj}_i \gamma(e)$  and  $\text{inj}_i \delta(\Upsilon(\Upsilon_k(m)))$  are values, it suffices to show  $(\text{inj}_i \gamma(e), \text{inj}_i \delta(\Upsilon(\Upsilon_k(m)))) \in \mathcal{V}_\zeta^+[\![s_1 + s_2]\!]_\delta$ .

By definition of  $\mathcal{V}_\zeta^+[\![s_1 + s_2]\!]_\delta$ , it suffices to show  $(\gamma(e), \delta(\Upsilon(\Upsilon_k(m)))) \in \mathcal{E}_\zeta^+[\![s_i]\!]_\delta$ , which follows by IH applied to  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash m : s_i^\dagger$ .

$$\text{Case (FD-Pair)} \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash m_1 : s_1^\dagger \uparrow e_1 \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash m_2 : s_2^\dagger \uparrow e_2}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \langle m_1, m_2 \rangle : s_1^\dagger \times s_2^\dagger \uparrow \langle e_1, e_2 \rangle}$$

This case is analogous to the previous case.

$$\text{Case (FD-Fun)} \quad \frac{\Sigma; \mathbf{G}_k; \Gamma^\dagger, x : s_1^\dagger \vdash m : s_2^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \lambda x: s_1^\dagger. m : s_1^\dagger \rightarrow s_2^\dagger \uparrow \lambda x: s_1. e}$$

For arbitrary  $\zeta, (\gamma, \Upsilon) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_\delta$ , we must show  $(\lambda x: s_1. \gamma(e), \lambda x: s_1^\dagger. \delta(\Upsilon(\Upsilon_k(m)))) \in \mathcal{E}_\zeta^+[\![s_1 \rightarrow s_2]\!]_\delta$ .  
Let  $e_2 = \gamma(e)$  and  $m_2 = \delta(\Upsilon(\Upsilon_k(m)))$ .

It suffices to show  $(\lambda x: s_1. e_2, \lambda x: s_1^\dagger. m_2) \in \mathcal{V}_\zeta^+[\![s_1 \rightarrow s_2]\!]_\delta$ .

For arbitrary  $(e_1, m_1) \in \mathcal{E}_\zeta^+[\![s_1]\!]_\delta$ , we must show  $(e_2[e_1/x], m_2[m_1/x]) \in \mathcal{E}_\zeta^+[\![s_2]\!]_\delta$ , which follows by IH applied to  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger, x : s_1^\dagger \vdash m : s_1^\dagger \rightarrow s_2^\dagger$ .

$$\text{Case (FD-Return)} \quad \frac{\Sigma; \mathbf{G}_k, k : (s^\dagger \rightarrow (T_\ell s)^\dagger); \Gamma^\dagger \vdash m[(T_\ell s)^\dagger/\beta] \langle \hat{\text{pf}}[\![\ell \leq T_\ell s]\!], k \rangle : T_\ell s^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \Lambda\beta::*. m : T_\ell s^\dagger \uparrow e}$$

For arbitrary  $\zeta, (\gamma, \Upsilon) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_\delta$ , we must show  $(\gamma(e), \delta(\Upsilon(\Upsilon_k(\Lambda\beta::*. m)))) \in \mathcal{E}_\zeta^+[\![T_\ell s^+]\!]_\delta$ .

Let  $e' = \gamma(e)$  and  $m' = \delta(\Upsilon(\Upsilon_k(m)))$ .

It suffices to show  $(e', (\Lambda\beta::*. m')) \in \mathcal{E}_\zeta^+[\![T_\ell s^+]\!]_\delta$ .

By IH applied to  $\Sigma; \mathbf{G}_k, k : (s^\dagger \rightarrow (T_\ell s)^\dagger); \Gamma^\dagger \vdash m[(T_\ell s)^\dagger/\beta] \langle \hat{\text{pf}}[\![\ell \leq T_\ell s]\!], k \rangle : T_\ell s^\dagger \uparrow e$  we know  $(e', m_1[(T_\ell s)^\dagger/\beta] \langle \hat{\text{pf}}[\![\ell \leq T_\ell s]\!], \eta_k^{\ell, s} \rangle) \in \mathcal{E}_\zeta^+[\![T_\ell s]\!]_\delta$ .

By Lemma 5.22 (Cross Lang. Relation Respects Target Relation), it suffices to show

$$(m_1[(T_\ell s)^\dagger/\beta] \langle \hat{\text{pf}}[\![\ell \leq T_\ell s]\!], \eta_k^{\ell, s} \rangle, \Lambda\beta::*. m_1) \in \mathcal{E}[\![\langle T_\ell s^+ \rangle^+]\!]_\rho^\Sigma.$$

By Lemma 5.18 (Free Theorem:  $\eta_k^{\ell, s}$  shuffling), we know  $(\Lambda\beta::*. m_1, m_1[(T_\ell s)^\dagger/\beta] \langle \hat{\text{pf}}[\![\ell \leq T_\ell s]\!], \eta_k^{\ell, s} \rangle) \in \mathcal{E}[\![\langle T_\ell s^+ \rangle^+]\!]_\rho^\Sigma$ .

By Lemma 5.7 (Target Relation Symmetric under Lattice Interp.), we know

$$(\mathbf{m}_1[(\mathbb{T}_\ell s)^\dagger/\beta] \langle \text{pf}[\ell \preceq \mathbb{T}_\ell s], \eta_k^{\ell, s} \rangle, \Lambda\beta::*\mathbf{m}_1) \in \mathcal{E}[\mathbb{T}_\ell s^\dagger]_\rho^\Sigma.$$

$$\text{Case (FD-Bind)} \frac{\begin{array}{c} \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_p : (\alpha_{\preceq} \alpha_\ell s^\dagger) \times (s'^\dagger \rightarrow s^\dagger) \\ \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : (\mathbb{T}_\ell s')^\dagger \uparrow e \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{prj}_2 \mathbf{m}_p : s'^\dagger \rightarrow s^\dagger \uparrow e' \end{array}}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} [s^\dagger] \mathbf{m}_p : s^\dagger \uparrow \text{bind } x = e \text{ in } e' x}$$

For arbitrary  $\zeta$ ,  $(\gamma, \gamma) \in \mathcal{G}_\zeta^+[\Gamma]_\delta$ , we must show  $(\gamma(\text{bind } x = e \text{ in } e' x), \delta(\gamma(\gamma_k(\mathbf{m} [s^\dagger] \mathbf{m}_{\text{pf}})))) \in \mathcal{E}_\zeta^+[\mathbb{s}]_\delta$ .

Note that since  $\Sigma_D; \Sigma_G, \mathbf{G}_k, \Gamma^\dagger \vdash \text{prj}_1 \mathbf{m}_{\text{pf}} : (\alpha_{\preceq} \alpha_\ell s^\dagger)$ , by Lemma 5.24 we know  $\ell \preceq s$ .

Let  $\text{bind } x = e_1 \text{ in } e_2 x = \gamma(\text{bind } x = e \text{ in } e' x)$  and  $\mathbf{m}_1 [s^\dagger] \mathbf{m}'_{\text{pf}} = \delta(\gamma(\gamma_k(\mathbf{m} [s^\dagger] \mathbf{m}_{\text{pf}})))$

By IH applied to  $\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : (\mathbb{T}_\ell s')^\dagger \uparrow e$ , we know  $e_1 \mapsto^* \eta_e e'_1$ , and  $\mathbf{m}_1 \mapsto^* \Lambda\beta::*\mathbf{m}'_1$ , and  $(\eta_e e'_1, \Lambda\beta::*\mathbf{m}'_1) \in \mathcal{V}_\zeta^+[\mathbb{T}_\ell s']_\delta$ .

Therefore,  $\exists \mathbf{m}'_1. (\mathbf{m}'_1[(\mathbb{T}_\ell s')^\dagger/\beta] \langle \text{pf}[\ell \preceq \mathbb{T}_\ell s'], \eta_k^{\ell, s} \rangle, \eta_k^{\ell, s} \mathbf{m}'_1) \in \mathcal{E}[(\mathbb{T}_\ell s')^\dagger]_\rho^\Sigma$  and  $(e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbb{s}']_\delta$ .

By Lemma 5.22 (Cross Lang. Relation Respects Target Relation) and Lemma 5.18 (Free Theorem:  $\eta_k^{\ell, s}$  shuffling), we know  $(\eta_e e'_1, \eta_k^{\ell, s} \mathbf{m}'_1) \in \mathcal{V}_\zeta^+[\mathbb{T}_\ell s']_\delta$ .

By Lemma 5.18 (Free Theorem:  $\eta_k^{\ell, s}$  shuffling) and Lemma 5.13 (Target Relation Transitivity) we know

$$((\eta_k^{\ell, s} \mathbf{m}'_1) [s^\dagger] \mathbf{m}'_{\text{pf}}, \mathbf{m}_1 [s^\dagger] \mathbf{m}'_{\text{pf}}) \in \mathcal{E}[\mathbb{s}^\dagger]_\rho^\Sigma.$$

Note that  $\text{bind } x = e_1 \text{ in } e_2 x \mapsto^* e_2 e'_1$ , and  $(\eta_k^{\ell, s} \mathbf{m}'_1) [s^\dagger] \mathbf{m}'_{\text{pf}} \mapsto^* (\text{prj}_2 \mathbf{m}'_{\text{pf}}) \mathbf{m}''$ .

Therefore, to show  $(\text{bind } x = e_1 \text{ in } e_2 x, \mathbf{m}_1 [s^\dagger] \mathbf{m}'_{\text{pf}}) \in \mathcal{E}_\zeta^+[\mathbb{s}]_\delta$ , it suffices to show  $(e_2 e'_1, \text{prj}_2 \mathbf{m}'_{\text{pf}} \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbb{s}]_\delta$ .

Since  $(e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+[\mathbb{s}']_\delta$ , it suffices to show  $(e_2, \text{prj}_2 \mathbf{m}'_{\text{pf}}) \in \mathcal{E}_\zeta^+[\mathbb{s}' \rightarrow \mathbb{s}]_\delta$ , which follows by IH applied to  $\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{prj}_2 \mathbf{m}_{\text{pf}} : s'^\dagger \rightarrow s^\dagger \uparrow e'$ .

$$\text{Case (FD-Subterm)} \frac{\begin{array}{c} \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F} : s_1^\dagger \Rightarrow s_2^\dagger \uparrow F' \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m} : s_1^\dagger \uparrow e \end{array}}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F}[\mathbf{m}] : s_2^\dagger \uparrow F'[e]}$$

By inlining the possible cases for  $\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{F} : s_1^\dagger \Rightarrow s_2^\dagger \uparrow F'$ , this case requires 3 subcases. Each follows simply by the IH.

$$\text{Case (FD-Val)} \frac{\mathbf{E}^\#[\mathbf{u}] \mapsto \mathbf{m}_1 \quad \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_1 : s_2^\dagger \uparrow e}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{E}^\#[\mathbf{u}] : s_2^\dagger \uparrow e}$$

Follows by IH applied to  $\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{m}_1 : s_2^\dagger \uparrow e$  and evaluation.

$$\text{Case (FD-Stack)} \frac{\begin{array}{c} \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 x_1. \mathbf{m}_1 \parallel \text{inj}_2 x_2. \mathbf{m}_2 : \mathbf{t} \text{ where } \exists s'. \mathbf{t} = s'^\dagger \\ \Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 y_1. \mathbf{E}^\#[\mathbf{m}_1[y_1/x_1]] \parallel \text{inj}_2 y_2. \mathbf{E}^\#[\mathbf{m}_2[y_2/x_2]] : s^\dagger \uparrow e \\ (\text{fresh } y'_1, y'_2) \end{array}}{\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \mathbf{E}^\#[\text{case } \mathbf{m} \text{ of } \text{inj}_1 x_1. \mathbf{m}_1 \parallel \text{inj}_2 x_2. \mathbf{m}_2] : s^\dagger \uparrow e}$$

Follows simply by IH applied to

$$\Sigma; \mathbf{G}_k; \Gamma^\dagger \vdash \text{case } \mathbf{m} \text{ of } \text{inj}_1 y_1. \mathbf{E}^\#[\mathbf{m}_1[y_1/x_1]] \parallel \text{inj}_2 y_2. \mathbf{E}^\#[\mathbf{m}_2[y_2/x_2]] : s^\dagger \uparrow e \text{ and evaluation.}$$

□

### Theorem 5.37 (Back-translation preserves semantics)

Let  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$ .

If  $\Sigma \vdash \mathbf{m} : s^\dagger$ , then  $\Sigma; \vdash; \mathbf{m} : s^\dagger \uparrow e$  and  $\forall \zeta. (e, \mathbf{m}) \in \mathcal{E}_\zeta^+[\mathbb{s}]_\delta$ .

### Proof

Follows from Lemma 5.36 and Theorem 5.35. □



## 5.6 Equivalence Preservation

**Lemma 5.38 (Preservation and Reflection of Closed Terms)**

Let  $\rho = \llbracket \mathcal{L}_\ell^+ \rrbracket_\zeta^\Sigma$ ,  $(e_1, \mathbf{m}_1) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s} \rrbracket_{\rho_1}$ , and  $(e_2, \mathbf{m}_2) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s} \rrbracket_{\rho_2}$ . Then:

1a If  $(e_1, e_2) \in \mathcal{E} \llbracket \mathbf{s} \rrbracket_\zeta$ , then  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E} \llbracket \mathbf{s}^+ \rrbracket_\rho^\Sigma$ .

1b If  $(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{E} \llbracket \mathbf{s}^+ \rrbracket_\rho^\Sigma$ , then  $(e_1, e_2) \in \mathcal{E} \llbracket \mathbf{s} \rrbracket_\zeta$ .

2a If  $(v_1, v_2) \in \mathcal{V} \llbracket \mathbf{s} \rrbracket_\zeta$ , then  $(u_1, u_2) \in \mathcal{V} \llbracket \mathbf{s}^+ \rrbracket_\rho^\Sigma$ .

2b If  $(u_1, u_2) \in \mathcal{V} \llbracket \mathbf{s}^+ \rrbracket_\rho^\Sigma$ , then  $(v_1, v_2) \in \mathcal{V} \llbracket \mathbf{s} \rrbracket_\zeta$ .

**Proof**

Part 1 follows easily from part 2.

Proof by induction on the structure of the type  $\mathbf{s}$ . There are two interesting cases.

**Case 2a  $\mathbf{s}_1 \rightarrow \mathbf{s}_2$**

(1) Have:  $(\lambda x : \mathbf{s}_1. e_2, \lambda x : \mathbf{s}_1. e'_2) \in \mathcal{V} \llbracket \mathbf{s}_1 \rightarrow \mathbf{s}_2 \rrbracket_\zeta$ .

(2) Have:  $(\lambda x : \mathbf{s}_1. e_2, \lambda x : \mathbf{s}_1^+. \mathbf{m}_2) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rightarrow \mathbf{s}_2 \rrbracket_{\rho_1}$ .

(3) Have:  $(\lambda x : \mathbf{s}_1. e'_2, \lambda x : \mathbf{s}_1^+. \mathbf{m}'_2) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rightarrow \mathbf{s}_2 \rrbracket_{\rho_2}$ .

Must show:  $(\lambda x : \mathbf{s}_1^+. \mathbf{m}_2, \lambda x : \mathbf{s}_1^+. \mathbf{m}'_2) \in \mathcal{E} \llbracket \mathbf{s}_1^+ \rightarrow \mathbf{s}_2^+ \rrbracket_\rho^\Sigma$

(4) Consider arbitrary  $\mathbf{m}_1, \mathbf{m}'_1$  such that  $(\mathbf{m}_1, \mathbf{m}'_1) \in \mathcal{E} \llbracket \mathbf{s}_1^+ \rrbracket_\rho^\Sigma$ .

Must show:  $(\mathbf{m}_2[\mathbf{m}_1/x], \mathbf{m}'_2[\mathbf{m}'_1/x]) \in \mathcal{E} \llbracket \mathbf{s}_2^+ \rrbracket_\rho^\Sigma$

(5) Have:  $\exists e_1. (e_1, \mathbf{m}_1) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rrbracket_{\rho_1}$  from Theorem 5.37. (Back-translation preserves semantics)

(6) Have:  $\exists e'_1. (e'_1, \mathbf{m}'_1) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rrbracket_{\rho_2}$  from Theorem 5.37 (Back-translation preserves semantics).

(7) Have:  $(e_1, e'_1) \in \mathcal{E} \llbracket \mathbf{s}_1 \rrbracket_\zeta$  by IH (1b) apply to  $\mathbf{s}_1$ .

Instantiating 1 with 7:  $(e_2[e_1/x], e'_2[e'_1/x]) \in \mathcal{E} \llbracket \mathbf{s}_2 \rrbracket_\zeta$

Instantiating 2 with 5:  $(e_2[e_1/x], \mathbf{m}_2[\mathbf{m}_1/x]) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_2 \rrbracket_{\rho_1}$

Instantiating 3 with 6:  $(e'_2[e'_1/x], \mathbf{m}'_2[\mathbf{m}'_1/x]) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_2 \rrbracket_{\rho_2}$

Proceed by induction (1a):  $(\mathbf{m}_2[\mathbf{m}_1/x], \mathbf{m}'_2[\mathbf{m}'_1/x]) \in \mathcal{E} \llbracket \mathbf{s}_2^+ \rrbracket_\rho^\Sigma$

**Case 2a  $\mathbf{T}_\ell \mathbf{s}_1$**

(1) Have:  $(\eta_\ell e_2, \eta_\ell e'_2) \in \mathcal{E} \llbracket \mathbf{T}_\ell \mathbf{s}_1 \rrbracket_\zeta$ .

(2) Have:  $(\eta_\ell e_2, \Lambda \beta :: *. \mathbf{m}_1) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{T}_\ell \mathbf{s}_1 \rrbracket_{\rho_1}$ .

(3) Have:  $(\eta_\ell e'_2, \Lambda \beta :: *. \mathbf{m}'_1) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{T}_\ell \mathbf{s}_1 \rrbracket_{\rho_2}$ .

Must show:  $(\Lambda \beta :: *. \mathbf{m}_1, \Lambda \beta :: *. \mathbf{m}'_1) \in \mathcal{V} \llbracket (\mathbf{T}_\ell \mathbf{s}_1)^+ \rrbracket_\rho^\Sigma$

To show this, there are two cases.

$\ell \sqsubseteq \zeta$  This case is indirect, *i.e.*, doesn't use the induction hypothesis.

Have:  $(e_2, e'_2) \in \mathcal{E} \llbracket \mathbf{s}_1 \rrbracket_\zeta$  from 1.

Have:  $\exists \mathbf{m}_2. (\mathbf{m}_1[(\mathbf{T}_\ell \mathbf{s}_1)^+/\beta] \langle \text{pf} \llbracket \ell \leq \mathbf{T}_\ell \mathbf{s} \rrbracket, \eta_k^{\ell, \mathbf{s}}, \eta_k^{\ell, \mathbf{s}} \mathbf{m}_2 \rangle) \in \mathcal{E} \llbracket (\mathbf{T}_\ell \mathbf{s}_1)^+ \rrbracket_\rho^\Sigma$  and  $(e_2, \mathbf{m}_2) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rrbracket_{\rho_1}$  from 2.

Have:  $\exists \mathbf{m}'_2. (\mathbf{m}'_1[(\mathbf{T}_\ell \mathbf{s}_1)^+/\beta] \langle \text{pf} \llbracket \ell \leq \mathbf{T}_\ell \mathbf{s} \rrbracket, \eta_k^{\ell, \mathbf{s}}, \eta_k^{\ell, \mathbf{s}} \mathbf{m}'_2 \rangle) \in \mathcal{E} \llbracket (\mathbf{T}_\ell \mathbf{s}_1)^+ \rrbracket_\rho^\Sigma$  and  $(e'_2, \mathbf{m}'_2) \in \mathcal{E}_\zeta^+ \llbracket \mathbf{s}_1 \rrbracket_{\rho_2}$  from 3.

Therefore:  $(\mathbf{m}_2, \mathbf{m}'_2) \in \mathcal{E} \llbracket \mathbf{s}_1^+ \rrbracket_\rho^\Sigma$ .

Have:  $(\Lambda\beta::*.m_1, \eta_k^{\ell,s} m_2) \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$  by Transitivity (Lemma 5.13) and  $\eta_k^{\ell,s}$  shuffling (Lemma 5.18).

Have:  $(\Lambda\beta::*.m'_1, \eta_k^{\ell,s} m'_2) \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$  by Transitivity (Lemma 5.13) and  $\eta_k^{\ell,s}$  shuffling (Lemma 5.18).

Have:  $(\eta_k^{\ell,s}, \eta_k^{\ell,s}) \in \mathcal{E}[s_1^+ \rightarrow (T_\ell s_1)^+]_{\rho}^{\Sigma}$  by Fundamental Property (Theorem 5.14).

Therefore:  $(\eta_k^{\ell,s} m_2, \eta_k^{\ell,s} m'_2) \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$ .

Have:  $(\Lambda\beta::*.m_1, \Lambda\beta::*.m'_1 \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma})$  by Transitivity (Lemma 5.13) and Symmetry (Lemma 5.7).

$\ell \not\sqsubseteq \zeta$  Consider arbitrary  $\pi$  such that  $\pi \in Rel^{\Sigma}$ .

Must show:  $(m_1[\pi_1/\beta], m'_1[\pi_2/\beta]) \in \mathcal{E}[(\alpha_{\leq} \alpha_{\ell} \beta) \times (s_1^+ \rightarrow \beta)]_{\rho[\alpha::\kappa \mapsto \pi]}^{\Sigma}$

Have:  $m_1[\pi_1/\beta] \mapsto^* \lambda x: (\alpha_{\leq} \alpha_{\ell} \pi_1) \times (s_1^+ \rightarrow \pi_1).m_2$ .

Have:  $m'_1[\pi_2/\beta] \mapsto^* \lambda x: (\alpha_{\leq} \alpha_{\ell} \pi_2) \times (s_1^+ \rightarrow \pi_2).m'_2$ .

Let  $\rho' = \rho[\alpha::\kappa \mapsto \pi]$ .

Consider arbitrary  $m, m'$  such that  $(m, m') \in \mathcal{E}[(\alpha_{\leq} \alpha_{\ell} \beta) \times (s_1^+ \rightarrow \beta)]_{\rho'}^{\Sigma}$ .

Must show:  $(m_2[m/x], m'_2[m'/x]) \in \mathcal{E}[\beta]_{\rho'}^{\Sigma}$

By definition of  $\rho$ , namely the relation on  $\alpha_{\leq}$ , we know  $\pi_R = Atom[\pi_1, \pi_2]^{\Sigma}$

Therefore:  $(m_2[m/x], m'_2[m'/x]) \in \mathcal{E}[\beta]_{\rho'}^{\Sigma}$ .

**Case 2b**  $T_\ell s_1$

(1) Have:  $(\Lambda\beta::*.m_1, \Lambda\beta::*.m'_1) \in \mathcal{V}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$ .

(2) Have:  $(\eta_{\ell} e_2, \Lambda\beta::*.m_1) \in \mathcal{E}_{\zeta}^+[(T_\ell s_1)]_{\rho_1}$ .

(3) Have:  $(\eta_{\ell} e'_2, \Lambda\beta::*.m'_1) \in \mathcal{E}_{\zeta}^+[(T_\ell s_1)]_{\rho_1}$ .

Must show:  $(\eta_{\ell} e_2, \eta_{\ell} e'_2) \in \mathcal{V}[(T_\ell s_1)]_{\zeta}$

Assume that  $\ell \sqsubseteq \zeta$ :

Must show:  $(e_2, e'_2) \in \mathcal{E}[s_1]_{\zeta}$

Let  $R = \{(m_1, m_2) \in Atom[(T_\ell s_1)^+]_{\rho}^{\Sigma} \mid$   
 $\ell \sqsubseteq \zeta \implies (\forall m', m''. ((m_1, \eta_k^{\ell,s} m') \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma} \wedge$   
 $(m_2, \eta_k^{\ell,s} m'') \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}) \implies$   
 $(m', m'') \in \mathcal{E}[s_1^+]_{\rho}^{\Sigma})\}$ .

Let  $\pi = (\rho_1((T_\ell s_1)^+), \rho_2((T_\ell s_1)^+), R)$ .

Instantiating 1 with  $\pi$ :

Have:  $m_1[\rho_1((T_\ell s_1)^+)/\beta] \mapsto^* \lambda x: (\alpha_{\leq} \alpha_{\ell} (T_\ell s_1)^+) \times (s^+ \rightarrow (T_\ell s_1)^+).m_2$ .

Have:  $m'_1[\rho_1((T_\ell s_1)^+)/\beta] \mapsto^* \lambda x: (\alpha_{\leq} \alpha_{\ell} (T_\ell s_1)^+) \times (s^+ \rightarrow (T_\ell s_1)^+).m'_2$ .

Let  $\rho' = \rho[\beta \mapsto \pi]$ .

(4) Have:  $(\lambda x: (\alpha_{\leq} \alpha_{\ell} (T_\ell s_1)^+) \times (s^+ \rightarrow (T_\ell s_1)^+).m_2, \lambda x: (\alpha_{\leq} \alpha_{\ell} (T_\ell s_1)^+) \times (s^+ \rightarrow (T_\ell s_1)^+).m'_2) \in \mathcal{E}[(\alpha_{\leq} \alpha_{\ell} \beta) \times (s^+ \rightarrow \beta)]_{\rho'}^{\Sigma}$ .

Let  $m = \langle pf[\ell \preceq T_\ell s_1], \eta_k^{\ell,s} \rangle$ .

Note that if  $\ell \not\sqsubseteq \zeta$ ,  $R = Atom[(T_\ell s_1)^+]_{\rho}^{\Sigma}$

(5) Therefore:  $(m, m) \in \mathcal{E}[(\alpha_{\leq} \alpha_{\ell} \beta) \times (s^+ \rightarrow \beta)]_{\rho}^{\Sigma}$ .

Have:  $(m_2[m/x], m'_2[m/x]) \in \mathcal{E}[\beta]_{\rho'}^{\Sigma}$  from 4 and 5.

(6) Have:  $\forall m', m''$  such that If  $(m_2[m/x], \eta_k^{\ell,s} m') \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$  and  $(m'_2[m/x], \eta_k^{\ell,s} m'') \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$ , then  $(m', m'') \in \mathcal{E}[s_1^+]_{\rho}^{\Sigma}$  by definition of  $R$  and by assumption  $\ell \sqsubseteq \zeta$ .

Have:  $\exists \hat{m}_2. (m_1[(T_\ell s_1)^+/\beta] m, \eta_k^{\ell,s} \hat{m}_2) \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$  and  $(e_2, \hat{m}_2) \in \mathcal{E}_{\zeta}^+[s_1]_{\rho_1}$  by 2.

Have:  $\exists \hat{\mathbf{m}}_2'. (\mathbf{m}_1' [(T_\ell s_1)^+ / \beta] \mathbf{m}, \eta_k^{\ell, s} \hat{\mathbf{m}}_2') \in \mathcal{E}[(T_\ell s_1)^+]_{\rho}^{\Sigma}$  and  $(e_2', \hat{\mathbf{m}}_2') \in \mathcal{E}_{\zeta}^+[[s_1]]_{\rho_2}$  by 3.  
Instantiating 6 with  $\hat{\mathbf{m}}_2$  and  $\hat{\mathbf{m}}_2'$ :  $(\hat{\mathbf{m}}_2, \hat{\mathbf{m}}_2') \in \mathcal{E}[[s_1]^+]_{\rho}^{\Sigma}$   
Proceed by induction (1b):  $(e_2, e_2') \in \mathcal{E}[[s_1]_{\zeta}]$

□

**Lemma 5.39 (Equivalence of Substitutions)**

Let  $\rho = [[\mathcal{L}_\ell^+]_{\zeta}^{\Sigma}]$ ,  $(\gamma_1, \gamma_1) \in \mathcal{G}_{\zeta}^+[[s]]_{\rho_1}$ , and  $(\gamma_2, \gamma_2) \in \mathcal{G}_{\zeta}^+[[s]]_{\rho_2}$ , then:

1. If  $(\gamma_1, \gamma_2) \in \mathcal{G}[[s]_{\zeta}]$ , then  $(\gamma_1, \gamma_2) \in \mathcal{G}[[s^+]_{\rho}^{\mathbf{D}; \mathbf{G}}]$
2. If  $(\gamma_1, \gamma_2) \in \mathcal{G}[[s^+]_{\rho}^{\mathbf{D}; \mathbf{G}}]$ , then  $(\gamma_1, \gamma_2) \in \mathcal{G}[[s]_{\zeta}]$

**Proof**

Follows from Part 1 of Lemma 5.38.

□

The following final theorem requires a few uninteresting extensions:

$\boxed{\vdash \gamma : \Gamma \rightsquigarrow \gamma}$  where  $\mathcal{L}_\ell^+; \mathcal{L}_{\zeta}^+; \preceq^+ \vdash \gamma : \Gamma^+$

$$\frac{}{\vdash \emptyset : \cdot \rightsquigarrow \emptyset} \qquad \frac{\cdot \vdash e : s \rightsquigarrow m \quad \vdash \gamma : \Gamma \rightsquigarrow \gamma}{\vdash \gamma[x \mapsto e] : \Gamma, x : s \rightsquigarrow \gamma[x \mapsto m]}$$

Figure 31: DCC to  $F_\omega$ : Subst. Translation

**Corollary 5.40 (Translation of Substitutions)**

Let  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$ ,  $\gamma_{\zeta} = [[\mathcal{L}_{\zeta}^+]]$ , and  $\gamma_{\preceq} = [[\preceq^+]]$ .

If  $\vdash \gamma : \Gamma$  then  $\vdash \gamma : \Gamma \rightsquigarrow \gamma$  and  $(\gamma, \gamma_{\zeta}(\gamma_{\preceq}(\gamma))) \in \mathcal{G}_{\zeta}^+[[\Gamma]]_{\delta}$

$\boxed{\Sigma; \mathbf{G}_k; \Gamma^{\dagger} \vdash \gamma : \Gamma^{\dagger} \uparrow \gamma}$  where  $\vdash \gamma : \Gamma$

$$\frac{}{\Sigma; \mathbf{G}_k; \Gamma^{\dagger} \vdash \emptyset : \cdot \uparrow \emptyset} \qquad \frac{\Sigma; \mathbf{G}_k; \Gamma^{\dagger} \mid \cdot \vdash m : s^{\dagger} \uparrow e \quad \Sigma; \mathbf{G}_k; \Gamma^{\dagger} \vdash \gamma : \Gamma^{\dagger} \uparrow \gamma}{\Sigma; \mathbf{G}_k; \Gamma^{\dagger} \vdash \gamma[x \mapsto m] : \Gamma^{\dagger}, x : s^{\dagger} \uparrow \gamma[x \mapsto e]}$$

Figure 32: Back-translation:  $F_\omega$  substitutions

**Corollary 5.41 (Back-translation of Substitutions)**

Let  $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$

If  $\Sigma \vdash \gamma : \Gamma^+$  then  $\exists \gamma. \Sigma; \mathbf{G}_k; \Gamma^{\dagger} \vdash \gamma : \Gamma^{\dagger} \uparrow \gamma$  and  $(\gamma, \gamma) \in \mathcal{G}_{\zeta}^+[[\Gamma]]_{\delta}$

**Theorem 5.42** ( $\approx_\zeta$  Preservation and Reflection)

Let  $\Gamma \vdash e_1 : s \rightsquigarrow \mathbf{m}_1$  and  $\Gamma \vdash e_2 : s \rightsquigarrow \mathbf{m}_2$ .

1. If  $\Gamma \vdash e_1 \approx_\zeta e_2 : s$ , then  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m}_1 \approx_\zeta \mathbf{m}_2 : s^+$ .
2. If  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m}_1 \approx_\zeta \mathbf{m}_2 : s^+$ , then  $\Gamma \vdash e_1 \approx_\zeta e_2 : s$ .

**Proof**

Part 1 follows from Lemma 5.23 (Correctness of Translation), Lemma 5.41 (Back-translation of Substitutions) and part 1a of Lemma 5.38 (Preservation and Reflection of Closed Terms).

Part 2 follows from Lemma 5.23 (Correctness of Translation), Lemma 5.40 (Translation of Substitutions), and part 1b of Lemma 5.38 (Preservation and Reflection of Closed Terms).  $\square$

**Theorem 5.43 (Noninterference: Indirect Proof)**

If  $\Gamma \vdash e : s$  then  $\Gamma \vdash e \approx_\zeta e : s$

**Proof**

By correctness of the translation:

$\Gamma \vdash e : s \rightsquigarrow \mathbf{m}$ ,  $\Gamma \vdash e \simeq \mathbf{m} : s$ , and  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : s^+$ .

By parametricity,  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx \mathbf{m} : s^+$

Since  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx \mathbf{m} : s^+$  quantifies over *all*  $\mathbf{D}$ ,  $\mathbf{G}$ ,  $\rho$ ,  $\gamma_1$ ,  $\gamma_2$ , the *particular* implementations we use in the observer-sensitive definition work, so:  $\mathcal{L}_\ell^+; \mathcal{L}_\perp^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx_\zeta \mathbf{m} : s^+$

By reflection,  $\Gamma \vdash e \approx_\zeta e : s$ .  $\square$

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