Noninterference for Free

(Technical Appendix)

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Contents

1 Source Language: DCC

This section describes the terminating portion of DCC [1], with slightly altered semantics as described by Tse and Zdancewic [3].

1.1 Syntax and Dynamic Semantics

DCC is a call-by-name, simply-typed λ-calculus, extended with a lattice of monads which allow dependencies to be expressed in a program. Except for the monadic operations, the language is completely standard. Figure 1 defines the syntax and dynamic semantics for DCC.

$$
\begin{array}{llcl}
\textit{Types} & \mathsf{s} & ::= & \mathsf{1} \mid \mathsf{s}_1 \times \mathsf{s}_2 \mid \mathsf{s}_1 + \mathsf{s}_2 \mid \mathsf{s}_1 \to \mathsf{s}_2 \mid \mathsf{T}_\ell\,\mathsf{s} \\
\textit{Values} & \mathsf{v} & ::= & \mathsf{x} \mid \langle\rangle \mid \langle \mathsf{e}_1, \mathsf{e}_2 \rangle \mid \mathsf{inj_i}\,\mathsf{e} \mid \lambda \mathsf{x} : \mathsf{s}.\, \mathsf{e} \mid \eta_\ell\,\mathsf{e} \\
\textit{Terms} & \mathsf{e} & ::= & \mathsf{v} \mid \mathsf{prj_i}\,\mathsf{e} \mid \mathsf{case}\,\mathsf{e}\,\mathsf{of}\,\mathsf{inj}_1\,\mathsf{x}_1.\,\mathsf{e}_1 \parallel \mathsf{inj}_2\,\mathsf{x}_2.\,\mathsf{e}_2 \mid \mathsf{bind}\,\mathsf{x} = \mathsf{e}_1\,\mathsf{in}\,\mathsf{e}_2 \mid \mathsf{e}_1\,\mathsf{e}_2 \\
\textit{Eval. Contexts} & \mathsf{E} & ::= & [\cdot]_{\mathsf{S}} \mid \mathsf{case}\,\mathsf{E}\,\mathsf{of}\,\mathsf{inj}_1\,\mathsf{x}_1.\,\mathsf{e}_1 \parallel \mathsf{inj}_2\,\mathsf{x}_2.\,\mathsf{e}_2 \mid \mathsf{bind}\,\mathsf{x} = \mathsf{E}\,\mathsf{in}\,\mathsf{e} \mid \mathsf{E}\,\mathsf{e} \\
\textit{Lattice} & \mathcal{L} & ::= & (\mathcal{L}_\ell, \mathcal{L}_\sqsubseteq) \\
\textit{Lattice Labels} & \mathcal{L}_\ell & ::= & \cdot \mid \mathcal{L}_\ell, \ell \\
\textit{Lattice Ordering} & \mathcal{L}_\sqsubseteq & ::= & \cdot \mid \mathcal{L}_\sqsubseteq, \ell_1 \sqsubseteq \ell_2
\end{array}
$$

$\boxed{\mathsf{e} \longmapsto \mathsf{e}'}$

$$
\begin{array}{rcl}
\mathsf{prj_i}\,\langle \mathsf{e}_1, \mathsf{e}_2 \rangle & \longmapsto & \mathsf{e_i} \\
\mathsf{case}\,(\mathsf{inj_i}\,\mathsf{e})\,\mathsf{of}\,\mathsf{inj}_1\,\mathsf{x}_1.\,\mathsf{e}_1 \parallel \mathsf{inj}_2\,\mathsf{x}_2.\,\mathsf{e}_2 & \longmapsto & \mathsf{e}_i[\mathsf{e}/\mathsf{x}_i] \\
(\lambda \mathsf{x} : \mathsf{s}.\, \mathsf{e}_1)\,\mathsf{e}_2 & \longmapsto & \mathsf{e}_1[\mathsf{e}_2/\mathsf{x}] \\
\mathsf{bind}\,\mathsf{x} = \eta_\ell\,\mathsf{e}_1\,\mathsf{in}\,\mathsf{e}_2 & \longmapsto & \mathsf{e}_2[\mathsf{e}_1/\mathsf{x}]
\end{array}
$$

$$
\frac{\mathsf{e} \longmapsto \mathsf{e}'}{\mathsf{E}[\mathsf{e}] \longmapsto \mathsf{E}[\mathsf{e}']}
$$

Figure 1: DCC: Syntax and Dynamic Semantics

1.2 Static Semantics

$\boxed{\ell \sqsubseteq \ell'}$ ℓ is at or below ℓ' in the lattice $\mathcal{L}$

$\boxed{\ell \preceq s}$ (type s is protected at ℓ)

$$\text{(P-Unit)}\ \frac{}{\ell \preceq 1} \qquad \text{(P-Pair)}\ \frac{\ell \preceq s_1 \qquad \ell \preceq s_2}{\ell \preceq s_1 \times s_2} \qquad \text{(P-Fun)}\ \frac{\ell \preceq s_2}{\ell \preceq s_1 \to s_2}$$

$$\text{(P-Label1)}\ \frac{\ell \not\sqsubseteq \ell' \qquad \ell \preceq s}{\ell \preceq T_{\ell'}\ s} \qquad \text{(P-Label2)}\ \frac{\ell \sqsubseteq \ell'}{\ell \preceq T_{\ell'}\ s}$$

Figure 2: DCC: Protection Semantics

$$\textit{Term Environment} \quad \Gamma \quad ::= \quad \cdot \mid \Gamma, x : s$$

$\boxed{\Gamma \vdash e : s}$

$$\text{(DT-Unit)}\ \frac{}{\Gamma \vdash \langle\rangle : 1} \qquad \text{(DT-Pair)}\ \frac{\Gamma \vdash e_1 : s_1 \qquad \Gamma \vdash e_2 : s_2}{\Gamma \vdash \langle e_1, e_2 \rangle : s_1 \times s_2} \qquad \text{(DT-Prj)}\ \frac{\Gamma \vdash e : s_1 \times s_2}{\Gamma \vdash \mathsf{prj_i}\ e : s_i}$$

$$\text{(DT-Sum)}\ \frac{\Gamma \vdash e : s_i}{\Gamma \vdash \mathsf{inj_i}\ e : s_1 + s_2} \qquad \text{(DT-Case)}\ \frac{\Gamma \vdash e : s_1 + s_2 \qquad \Gamma, x_i : s_i \vdash e_i : s}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ \mathsf{inj_1}\ x_1.\ e_1 \parallel \mathsf{inj_2}\ x_2.\ e_2 : s}$$

$$\text{(DT-Var)}\ \frac{(x : s) \in \Gamma}{\Gamma \vdash x : s} \qquad \text{(DT-Fun)}\ \frac{\Gamma, x : s_1 \vdash e : s_2}{\Gamma \vdash \lambda x : s_1.\ e : s_1 \to s_2} \qquad \text{(DT-App)}\ \frac{\Gamma \vdash e_1 : s_1 \to s_2 \qquad \Gamma \vdash e_2 : s_1}{\Gamma \vdash e_1\ e_2 : s_2}$$

$$\text{(DT-Prot)}\ \frac{\Gamma \vdash e : s}{\Gamma \vdash \eta_\ell\ e : T_\ell\ s} \qquad \text{(DT-Bind)}\ \frac{\Gamma \vdash e_1 : T_\ell\ s_1 \qquad \Gamma, x : s_1 \vdash e_2 : s_2 \qquad \ell \preceq s_2}{\Gamma \vdash \mathsf{bind}\ x = e_1\ \mathsf{in}\ e_2 : s_2}$$

Figure 3: DCC: Typing Rules

1.3 DCC Logical Relation

Figure 4 presents the logical relation for DCC. We define a relation, $\mathcal{V}[\![\mathsf{s}]\!]_\zeta$, which relates closed values at type s, a relation $\mathcal{E}[\![\mathsf{s}]\!]_\zeta$, which relates closed expressions, and finally extend it to open terms (written $\Gamma \vdash \mathsf{e} \approx_\zeta \mathsf{e}' : \mathsf{s}$). Note that each relation is parameterized by an observer ζ.

$$
\begin{array}{rcl}
\mathit{Atom}\,[\mathsf{s}] & = & \{\, (\mathsf{e}_1, \mathsf{e}_2) \mid\ \vdash \mathsf{e}_1 : \mathsf{s} \ \wedge\ \vdash \mathsf{e}_2 : \mathsf{s} \,\} \\
\mathit{Atom}^{\mathrm{val}}\,[\mathsf{s}] & = & \{\, (\mathsf{v}_1, \mathsf{v}_2) \mid\ \vdash \mathsf{v}_1 : \mathsf{s} \ \wedge\ \vdash \mathsf{v}_2 : \mathsf{s} \,\} \\
\\
\mathcal{V}[\![\mathsf{1}]\!]_\zeta & = & \{\, (\langle\rangle, \langle\rangle) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{1}] \,\} \\
\mathcal{V}[\![\mathsf{s} \times \mathsf{s}']\!]_\zeta & = & \{(\langle \mathsf{e}_1, \mathsf{e}_1' \rangle, \langle \mathsf{e}_2, \mathsf{e}_2' \rangle) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{s} \times \mathsf{s}'] \mid \\
 & & \quad (\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta \ \wedge\ (\mathsf{e}_1', \mathsf{e}_2') \in \mathcal{E}[\![\mathsf{s}']\!]_\zeta\} \\
\mathcal{V}[\![\mathsf{s} + \mathsf{s}']\!]_\zeta & = & \{(\mathsf{inj}_1\, \mathsf{e}_1, \mathsf{inj}_1\, \mathsf{e}_2) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{s} + \mathsf{s}'] \mid\ (\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta\} \\
 & \cup & \{(\mathsf{inj}_2\, \mathsf{e}_1, \mathsf{inj}_2\, \mathsf{e}_2) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{s} + \mathsf{s}'] \mid\ (\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}']\!]_\zeta\} \\
\mathcal{V}[\![\mathsf{s}' \to \mathsf{s}]\!]_\zeta & = & \{(\lambda \mathsf{x} : \mathsf{s}'.\, \mathsf{e}_1, \lambda \mathsf{x} : \mathsf{s}'.\, \mathsf{e}_2) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{s}' \to \mathsf{s}] \mid \\
 & & \quad \forall (\mathsf{e}_1', \mathsf{e}_2') \in \mathcal{E}[\![\mathsf{s}']\!]_\zeta.\ (\mathsf{e}_1[\mathsf{e}_1'/\mathsf{x}], \mathsf{e}_2[\mathsf{e}_2'/\mathsf{x}]) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta\} \\
\mathcal{V}[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_\zeta & = & \{(\eta_\ell\, \mathsf{e}_1, \eta_\ell\, \mathsf{e}_2) \in \mathit{Atom}^{\mathrm{val}}\,[\mathsf{T}_\ell\, \mathsf{s}] \mid\ \ell \sqsubseteq \zeta \implies (\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta\} \\
\\
\mathcal{E}[\![\mathsf{s}]\!]_\zeta & = & \{(\mathsf{e}_1, \mathsf{e}_2) \in \mathit{Atom}\,[\mathsf{s}] \mid\ \exists \mathsf{v}_1, \mathsf{v}_2.\ \mathsf{e}_1 \longmapsto^* \mathsf{v}_1 \ \wedge\ \mathsf{e}_2 \longmapsto^* \mathsf{v}_2 \ \wedge\ (\mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{s}]\!]_\zeta\} \\
\\
\mathcal{G}[\![\cdot]\!]_\zeta & = & (\emptyset, \emptyset) \\
\mathcal{G}[\![\Gamma, \mathsf{x} : \mathsf{s}]\!]_\zeta & = & \{(\gamma_1\,[\mathsf{x} \mapsto \mathsf{e}_1]\,, \gamma_2\,[\mathsf{x} \mapsto \mathsf{e}_2]) \mid\ (\gamma_1, \gamma_2) \in \mathcal{G}[\![\Gamma]\!]_\zeta \ \wedge\ ((\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta)\} \\
\\
\Gamma \vdash \mathsf{e}_1 \approx_\zeta \mathsf{e}_2 : \mathsf{s} & \stackrel{\mathrm{def}}{=} & \Gamma \vdash \mathsf{e}_1 : \mathsf{s} \ \wedge\ \Gamma \vdash \mathsf{e}_2 : \mathsf{s} \ \wedge \\
 & & \forall (\gamma_1, \gamma_2) \in \mathcal{G}[\![\Gamma]\!]_\zeta.\ (\gamma_1(\mathsf{e}_1), \gamma_2(\mathsf{e}_2)) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta
\end{array}
$$

Figure 4: DCC: Logical Relation

2 Target Language: F_ω

In this section we present the target language F_ω, the higher-order polymorphic lambda calculus with unit, pairs, and sums. The language is completely standard (see Pierce [2], Chapter 30).

2.1 Syntax and Dynamic Semantics

$$
\begin{array}{llll}
\textit{Kinds} & \kappa & ::= & * \mid \kappa \to \kappa \\
\textit{Types} & \mathbf{t} & ::= & \mathbf{1} \mid \mathbf{t}_1 \times \mathbf{t}_2 \mid \mathbf{t}_1 + \mathbf{t}_2 \mid \mathbf{t}_1 \to \mathbf{t}_2 \mid \alpha \mid \forall\alpha{::}\kappa.\,\mathbf{t} \mid \lambda\alpha{::}\kappa.\mathbf{t} \mid \mathbf{t}_1\,\mathbf{t}_2 \\
\textit{Values} & \mathbf{u} & ::= & \mathbf{x} \mid \langle\rangle \mid \langle \mathbf{m}_1, \mathbf{m}_2\rangle \mid \mathbf{inj_i}\ \mathbf{m} \mid \lambda\mathbf{x}{:}\mathbf{t}.\mathbf{m} \mid \Lambda\alpha{::}\kappa.\mathbf{m} \\
\textit{Terms} & \mathbf{m} & ::= & \mathbf{u} \mid \mathbf{prj_i}\ \mathbf{m} \mid \mathbf{case}\,\mathbf{m}\,\mathbf{of}\ \mathbf{inj}_1\ \mathbf{x}_1.\,\mathbf{m}_1 \parallel \mathbf{inj}_2\ \mathbf{x}_2.\ \mathbf{m}_2 \mid \mathbf{m}_1\,\mathbf{m}_2 \mid \mathbf{m}\,[\mathbf{t}] \\
\textit{Eval. Contexts} & \mathbf{E} & ::= & [\cdot]_{\mathbf{T}} \mid \mathbf{prj_i}\ \mathbf{E} \mid \mathbf{case}\,\mathbf{E}\,\mathbf{of}\ \mathbf{inj}_1\ \mathbf{x}_1.\,\mathbf{m}_1 \parallel \mathbf{inj}_2\ \mathbf{x}_2.\ \mathbf{m}_2 \mid \mathbf{E}\ \mathbf{m} \mid \mathbf{E}\,[\mathbf{t}]
\end{array}
$$

$\boxed{\mathbf{m} \longmapsto \mathbf{m}'}$

$$
\begin{array}{rcl}
\mathbf{prj_i}\ \langle \mathbf{m}_1, \mathbf{m}_2\rangle & \longmapsto & \mathbf{m_i} \\
\mathbf{case}\,(\mathbf{inj_i}\ \mathbf{m})\,\mathbf{of}\ \mathbf{inj}_1\ \mathbf{y}_1.\,\mathbf{m}_1 \parallel \mathbf{inj}_2\ \mathbf{y}_2.\ \mathbf{m}_2 & \longmapsto & \mathbf{m}_i[\mathbf{m}/\mathbf{y}_i] \\
(\lambda\mathbf{x}{:}\mathbf{t}.\mathbf{m}_1)\ \mathbf{m}_2 & \longmapsto & \mathbf{m}_1[\mathbf{m}_2/\mathbf{x}] \\
(\Lambda\alpha{::}\kappa.\mathbf{m})\,[\mathbf{t}] & \longmapsto & \mathbf{m}[\mathbf{t}/\alpha]
\end{array}
$$

$$
\frac{\mathbf{m} \longmapsto \mathbf{m}'}{\mathbf{E}[\mathbf{m}] \longmapsto \mathbf{E}[\mathbf{m}']}
$$

Figure 5: F_ω: Syntax and Dynamic Semantics

2.2 Static Semantics

$\boxed{\Delta \vdash t :: \kappa}$

$$\text{(FK-Var)}\ \frac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa} \qquad \text{(FK-Unit)}\ \frac{}{\Delta \vdash 1 :: *} \qquad \text{(FK-Pair)}\ \frac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \times t_2 :: *}$$

$$\text{(FK-Sum)}\ \frac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *} \qquad \text{(FK-Arrow)}\ \frac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \rightarrow t_2 :: *} \qquad \text{(FK-Abs)}\ \frac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha{::}\kappa.\, t :: *}$$

$$\text{(FK-Fun)}\ \frac{\Delta, \alpha :: \kappa_1 \vdash t :: \kappa_2}{\Delta \vdash \lambda \alpha{::}\kappa.t :: \kappa_1 \rightarrow \kappa_2} \qquad \text{(FK-App)}\ \frac{\Delta \vdash t_1 :: \kappa_1 \rightarrow \kappa_2 \qquad \Delta \vdash t_2 :: \kappa_1}{\Delta \vdash t_1\ t_2 :: \kappa_2}$$

$\boxed{\Delta \vdash \Gamma}$

$$\text{(FWF-Emp)}\ \frac{}{\Delta \vdash \cdot} \qquad \text{(FWF-Var)}\ \frac{\Delta \vdash \Gamma \qquad \Delta \vdash t :: *}{\Delta \vdash \Gamma, x : t}$$

Figure 6: F_ω: Kinding

$\boxed{t \equiv t'}$

$$\text{(FE-Refl)}\ \frac{}{t \equiv t} \qquad \text{(FE-Sym)}\ \frac{t_1 \equiv t_2}{t_2 \equiv t_1} \qquad \text{(FE-Trans)}\ \frac{t_1 \equiv t_2 \qquad t_2 \equiv t_3}{t_1 \equiv t_3}$$

$$\text{(FE-Pair)}\ \frac{t_1 \equiv t_1' \qquad t_2 \equiv t_2'}{t_1 \times t_2 \equiv t_1' \times t_2'} \qquad \text{(FE-Sum)}\ \frac{t_1 \equiv t_1' \qquad t_2 \equiv t_2'}{t_1 + t_2 \equiv t_1' + t_2'}$$

$$\text{(FE-Arrow)}\ \frac{t_1 \equiv t_1' \qquad t_2 \equiv t_2'}{t_1 \rightarrow t_2 \equiv t_1' \rightarrow t_2'} \qquad \text{(FE-All)}\ \frac{t_1 \equiv t_2}{\forall \alpha{::}\kappa.\, t_1 \equiv \forall \alpha{::}\kappa.\, t_2}$$

$$\text{(FE-Abs)}\ \frac{t_1 \equiv t_2}{\lambda \alpha{::}\kappa.t_1 \equiv \lambda \alpha{::}\kappa.t_2} \qquad \text{(FE-App)}\ \frac{t_1 \equiv t_1' \qquad t_2 \equiv t_2'}{t_1\ t_2 \equiv t_1'\ t_2'} \qquad \text{(FE-Beta)}\ \frac{}{(\lambda \alpha{::}\kappa.t_1)\ t_2 \equiv t_1[t_2/\alpha]}$$

Figure 7: F_ω: Type Equivalence

$$\begin{array}{llll} \textit{Type Environment} & \boldsymbol{\Delta} & ::= & \cdot \mid \boldsymbol{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa} \\ \textit{Term Environment} & \boldsymbol{\Gamma} & ::= & \cdot \mid \boldsymbol{\Gamma}, \mathbf{x} : \mathbf{t} \end{array}$$

$\boxed{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}}$

$$\text{(FT-Unit)}\ \frac{\boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \langle\rangle : \mathbf{1}} \qquad \text{(FT-Var)}\ \frac{\mathbf{x} : \mathbf{t} \in \boldsymbol{\Gamma} \qquad \boldsymbol{\Delta} \vdash \boldsymbol{\Gamma}}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{x} : \mathbf{t}} \qquad \text{(FT-Pair)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \qquad \boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_2}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \langle \mathbf{m_1}, \mathbf{m_2} \rangle : \mathbf{t}_1 \times \mathbf{t}_2}$$

$$\text{(FT-Prj)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{prj_i}\ \mathbf{m} : \mathbf{t}_i}$$

$$\text{(FT-Sum)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_i}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{inj_i}\ \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2}$$

$$\text{(FT-Case)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2 \qquad \boldsymbol{\Delta}; \boldsymbol{\Gamma}, \mathbf{x} : \mathbf{t}_1 \vdash \mathbf{m_1} : \mathbf{t} \qquad \boldsymbol{\Delta}; \boldsymbol{\Gamma}, \mathbf{x} : \mathbf{t}_2 \vdash \mathbf{m_2} : \mathbf{t}}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{case}\,\mathbf{m}\,\mathbf{of}\,\mathbf{inj}_1\ \mathbf{x}.\,\mathbf{m_1} \parallel \mathbf{inj}_2\ \mathbf{x}.\ \mathbf{m_2} : \mathbf{t}}$$

$$\text{(FT-Fun)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma}, \mathbf{x} : \mathbf{t}_1 \vdash \mathbf{m} : \mathbf{t}_2 \qquad \boldsymbol{\Delta} \vdash \mathbf{t}_1 :: *}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \boldsymbol{\lambda}\mathbf{x}{:}\mathbf{t}_1.\mathbf{m} : \mathbf{t}_1 \to \mathbf{t}_2} \qquad \text{(FT-App)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \to \mathbf{t}_2 \qquad \boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_1}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m_1}\ \mathbf{m_2} : \mathbf{t}_2}$$

$$\text{(FT-Abs)}\ \frac{\boldsymbol{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m} : \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}} \qquad \text{(FT-Inst)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}_1 \qquad \boldsymbol{\Delta} \vdash \mathbf{t}_2 :: \boldsymbol{\kappa}}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m}\,[\mathbf{t}_2] : \mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}]}$$

$$\text{(FT-Eqv)}\ \frac{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \qquad \mathbf{t}_1 \equiv \mathbf{t}_2 \qquad \boldsymbol{\Delta} \vdash \mathbf{t}_2 :: *}{\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_2}$$

Figure 8: F_ω: Typing

2.3 F_ω Logical Relation

In this subsection we present an open logical relation for F_ω.

$$
\begin{array}{rcl}
\mathit{Atom}\,[\mathbf{t}_1, \mathbf{t}_2]^{\mathbf{D};\mathbf{G}} & = & \{\,(\mathbf{m_1}, \mathbf{m_2}) \mid\ \mathbf{D} \vdash \mathbf{t}_1\ \wedge\ \mathbf{D} \vdash \mathbf{t}_2\ \wedge\ \mathbf{D};\mathbf{G} \vdash \mathbf{m_1} : \mathbf{t}_1\ \wedge\ \mathbf{D};\mathbf{G} \vdash \mathbf{m_2} : \mathbf{t}_2\,\} \\
\mathit{Atom}\,[\mathbf{t}]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} & = & \mathit{Atom}\,[\boldsymbol{\rho_1}(\mathbf{t}), \boldsymbol{\rho_2}(\mathbf{t})]^{\mathbf{D};\mathbf{G}} \\
\mathit{Rel}_{*}^{\mathbf{D};\mathbf{G}} & = & \{\,(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}) \mid \mathbf{R} \subseteq \mathit{Atom}\,[\mathbf{t}_1, \mathbf{t}_2]^{\mathbf{D};\mathbf{G}}\} \\
\mathit{Rel}_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}}^{\mathbf{D};\mathbf{G}} & = & \{(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}) \mid \\
& & (\forall \boldsymbol{\pi} \in \mathit{Rel}_{\boldsymbol{\kappa_1}}^{\mathbf{D};\mathbf{G}}.\ (\mathbf{t}_1\ \boldsymbol{\pi_1},\ \mathbf{t}_2\ \boldsymbol{\pi_2},\ (\mathbf{R}\ \boldsymbol{\pi})) \in \mathit{Rel}_{\boldsymbol{\kappa_2}}^{\mathbf{D};\mathbf{G}} \wedge \\
& & \quad (\forall \boldsymbol{\pi}' \in \mathit{Rel}_{\boldsymbol{\kappa_1}}^{\mathbf{D};\mathbf{G}}.\ \boldsymbol{\pi} \equiv_{\boldsymbol{\kappa_1}}^{\mathbf{D};\mathbf{G}} \boldsymbol{\pi}' \implies \\
& & \qquad \mathbf{R}\ \boldsymbol{\pi} \equiv_{\boldsymbol{\kappa_2}}^{\mathbf{D};\mathbf{G}} \mathbf{R}\ \boldsymbol{\pi}'\}
\end{array}
$$

$$
\begin{array}{lcl}
\boldsymbol{\rho} \equiv_{\boldsymbol{\kappa}}^{\mathbf{D};\mathbf{G}} \boldsymbol{\rho}' & \stackrel{\text{def}}{=} & \forall \boldsymbol{\alpha} :: \boldsymbol{\kappa} \in \mathrm{dom}(\boldsymbol{\rho}).\boldsymbol{\rho}(\boldsymbol{\alpha}) \equiv_{\boldsymbol{\kappa}}^{\mathbf{D};\mathbf{G}} \boldsymbol{\rho}'(\boldsymbol{\alpha}) \\
\boldsymbol{\pi} \equiv_{\boldsymbol{\kappa}}^{\mathbf{D};\mathbf{G}} \boldsymbol{\pi} & \stackrel{\text{def}}{=} & \boldsymbol{\pi_1} \equiv \boldsymbol{\pi}_1'\ \wedge\ \boldsymbol{\pi_2} \equiv \boldsymbol{\pi}_2'\ \wedge\ \boldsymbol{\pi}_{\mathbf{R}} \equiv_{\boldsymbol{\kappa}}^{\mathbf{D};\mathbf{G}} \boldsymbol{\pi}_{\mathbf{R}}' \\
\mathbf{R} \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathbf{R}' & \stackrel{\text{def}}{=} & \forall \mathbf{m_1}, \mathbf{m_2}.\ (\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R} \iff (\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R}' \\
\mathbf{R} \equiv_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}}^{\mathbf{D};\mathbf{G}} \mathbf{R}' & \stackrel{\text{def}}{=} & \forall \boldsymbol{\pi} \in \mathit{Rel}_{\boldsymbol{\kappa_1}}^{\mathbf{D};\mathbf{G}}.\ \mathbf{R}\ \boldsymbol{\pi} \equiv_{\boldsymbol{\kappa_2}}^{\mathbf{D};\mathbf{G}} \mathbf{R}'\ \boldsymbol{\pi}
\end{array}
$$

$$
\begin{array}{lcl}
\boldsymbol{\mathcal{T}}[\![\mathbf{t} :: *]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} & = & \boldsymbol{\mathcal{V}}[\![\mathbf{t}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} \\
\boldsymbol{\mathcal{T}}[\![\boldsymbol{\alpha} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} & = & \boldsymbol{\rho}_{\mathbf{R}}(\boldsymbol{\alpha}) \\
\boldsymbol{\mathcal{T}}[\![\boldsymbol{\lambda\alpha}{::}\boldsymbol{\kappa_1}.\mathbf{t} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} & = & \lambda_R \boldsymbol{\pi}.\{\boldsymbol{\mathcal{T}}[\![\mathbf{t} :: \boldsymbol{\kappa_2}]\!]_{\boldsymbol{\rho}[\boldsymbol{\alpha}::\boldsymbol{\kappa_1} \mapsto \boldsymbol{\pi}]}^{\mathbf{D};\mathbf{G}}\} \\
\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1\ \mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} & = & (\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}} \\
& & \quad (\boldsymbol{\rho_1}\ \mathbf{t}_2, \boldsymbol{\rho_2}\ \mathbf{t}_2, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_1}]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}}))
\end{array}
$$

Figure 9: F_ω: Logical Relations (Higher-Order)

$$
\begin{array}{rcl}
Atom^{\mathrm{val}}\,[\mathbf{t}_1,\mathbf{t}_2]^{\mathbf{D};\mathbf{G}} & = & \{\,(\mathbf{u}_1,\mathbf{u}_2) \mid\ \mathbf{D}\vdash\mathbf{t}_1 \ \wedge\ \mathbf{D}\vdash\mathbf{t}_2 \ \wedge\ \mathbf{D};\mathbf{G}\vdash\mathbf{u}_1:\mathbf{t}_1 \ \wedge\ \mathbf{D};\mathbf{G}\vdash\mathbf{u}_2:\mathbf{t}_2\,\} \\
Atom^{\mathrm{val}}\,[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}} & = & Atom^{\mathrm{val}}\,[\rho_1(\mathbf{t}),\rho_2(\mathbf{t})]^{\mathbf{D};\mathbf{G}} \\
\mathcal{V}[\![\alpha]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\mathbf{m}_1,\mathbf{m}_2)\in\rho_{\mathbf{R}}(\alpha)\} \\
\mathcal{V}[\![\mathbf{1}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{\,(\langle\rangle,\langle\rangle)\in Atom^{\mathrm{val}}\,[\mathbf{1}]_{\rho}^{\mathbf{D};\mathbf{G}}\,\} \\
\mathcal{V}[\![\mathbf{t}\times\mathbf{t}']\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\langle\mathbf{m}_1,\mathbf{m}_1'\rangle,\langle\mathbf{m}_2,\mathbf{m}_2'\rangle)\in Atom^{\mathrm{val}}\,[\mathbf{t}\times\mathbf{t}']_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
 & & \quad (\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} \ \wedge\ (\mathbf{m}_1',\mathbf{m}_2')\in\mathcal{E}[\![\mathbf{t}']\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
\mathcal{V}[\![\mathbf{t}+\mathbf{t}']\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\mathbf{inj}_1\ \mathbf{m}_1,\mathbf{inj}_1\ \mathbf{m}_2)\in Atom^{\mathrm{val}}\,[\mathbf{t}+\mathbf{t}']_{\rho}^{\mathbf{D};\mathbf{G}} \mid\ (\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
 & \cup & \{(\mathbf{inj}_2\ \mathbf{m}_1,\mathbf{inj}_2\ \mathbf{m}_2)\in Atom^{\mathrm{val}}\,[\mathbf{t}+\mathbf{t}']_{\rho}^{\mathbf{D};\mathbf{G}} \mid\ (\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{E}[\![\mathbf{t}']\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
\mathcal{V}[\![\mathbf{t}'\to\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\lambda\mathbf{x}{:}\rho_1(\mathbf{t}').\mathbf{m}_1,\lambda\mathbf{x}{:}\rho_2(\mathbf{t}').\mathbf{m}_2)\in Atom^{\mathrm{val}}\,[\mathbf{t}'\to\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
 & & \quad \forall(\mathbf{m}_1',\mathbf{m}_2')\in\mathcal{E}[\![\mathbf{t}']\!]_{\rho}^{\mathbf{D};\mathbf{G}}.\ (\mathbf{m}_1[\mathbf{m}_1'/\mathbf{x}],\mathbf{m}_2[\mathbf{m}_2'/\mathbf{x}])\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
\mathcal{V}[\![\forall\alpha{::}\kappa.\,\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\Lambda\alpha{::}\kappa.\mathbf{m}_1,\Lambda\alpha{::}\kappa.\mathbf{m}_2)\in Atom^{\mathrm{val}}\,[\forall\alpha{::}\kappa.\,\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
 & & \quad \forall\pi\in Rel_{\kappa}^{\mathbf{D};\mathbf{G}}. \\
 & & \qquad (\mathbf{m}_1[\pi_1/\alpha],\mathbf{m}_2[\pi_2/\alpha])\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho[\alpha::\kappa\mapsto\pi]}^{\mathbf{D};\mathbf{G}}\} \\
\mathcal{V}[\![\mathbf{t}_1\ \mathbf{t}_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \mathcal{T}[\![\mathbf{t}_1\ \mathbf{t}_2 :: *]\!]_{\rho}^{\mathbf{D};\mathbf{G}} \\
\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\mathbf{m}_1,\mathbf{m}_2)\in Atom\,[\mathbf{t}]_{\rho}^{\mathbf{D};\mathbf{G}} \mid \\
 & & \quad \exists\mathbf{m}_1',\mathbf{m}_2'.\ \mathbf{m}_1\longmapsto^{*}\mathbf{m}_1' \ \wedge\ \mathbf{m}_2\longmapsto^{*}\mathbf{m}_2' \ \wedge \\
 & & \quad \mathrm{irred}(\mathbf{m}_1') \ \wedge\ \mathrm{irred}(\mathbf{m}_2') \ \wedge \\
 & & \quad (\mathbf{m}_1',\mathbf{m}_2')\in\mathcal{V}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
\mathcal{D}[\![\cdot]\!]^{\mathbf{D};\mathbf{G}} & = & \emptyset \\
\mathcal{D}[\![\Delta,\alpha::\kappa]\!]^{\mathbf{D};\mathbf{G}} & = & \{\,\rho\,[\alpha::\kappa\mapsto\pi] \mid\ \rho\in\mathcal{D}[\![\Delta]\!]^{\mathbf{D};\mathbf{G}}\wedge\pi\in Rel_{\kappa}^{\mathbf{D};\mathbf{G}}\,\} \\
\mathcal{G}[\![\cdot]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & (\emptyset,\emptyset) \\
\mathcal{G}[\![\Gamma,\mathbf{x}:\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}} & = & \{(\gamma_1\,[\mathbf{x}\mapsto\mathbf{m}_1]\,,\gamma_2\,[\mathbf{x}\mapsto\mathbf{m}_2]) \mid \\
 & & \quad (\gamma_1,\gamma_2)\in\mathcal{G}[\![\Gamma]\!]_{\rho}^{\mathbf{D};\mathbf{G}} \ \wedge\ (\mathbf{m}_1,\mathbf{m}_2)\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\} \\
\Delta;\Gamma\vdash\mathbf{m}_1\approx\mathbf{m}_2:\mathbf{t} & \stackrel{\mathrm{def}}{=} & \Delta;\Gamma\vdash\mathbf{m}_1:\mathbf{t} \ \wedge\ \Delta;\Gamma\vdash\mathbf{m}_2:\mathbf{t} \ \wedge \\
 & & \forall\mathbf{D},\mathbf{G},\rho,\gamma_1,\gamma_2.\ \mathrm{dom}(\mathbf{D})\#\mathrm{dom}(\Delta) \ \wedge\ \mathrm{dom}(\mathbf{G})\#\mathrm{dom}(\Gamma) \ \wedge \\
 & & \rho\in\mathcal{D}[\![\Delta]\!]^{\mathbf{D};\mathbf{G}}\wedge(\gamma_1,\gamma_2)\in\mathcal{G}[\![\Gamma]\!]_{\rho}^{\mathbf{D};\mathbf{G}} \implies \\
 & & \quad (\rho_1\gamma_1(\mathbf{m}_1),\rho_2\gamma_2(\mathbf{m}_2))\in\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}
\end{array}
$$

Figure 10: F_ω: Logical Relation

3 Translation: DCC to F_ω

In this section, we present the type-directed translation from DCC to F_ω.

$$
\begin{array}{rcl}
\mathcal{L}_\ell^+ & = & \{\boldsymbol{\alpha}_\ell :: * \mid \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_\preceq :: * \to * \to *\} \\
\mathcal{L}_\sqsubseteq^+ & = & \{\mathbf{c}_{\ell'\ell} : \boldsymbol{\alpha}_{\ell'} \to \boldsymbol{\alpha}_\ell \mid \ell \sqsubseteq \ell' \in \mathcal{L}_\sqsubseteq\}
\end{array}
$$

$\boxed{\mathsf{s}^+}$ where $\mathcal{L}_\ell^+ \vdash \mathsf{s}^+ :: *$

$$
\begin{array}{rcl}
1^+ & = & \mathbf{1} \\
(\mathsf{s}_1 \times \mathsf{s}_2)^+ & = & \mathsf{s}_1^+ \times \mathsf{s}_2^+ \\
(\mathsf{s}_1 + \mathsf{s}_2)^+ & = & \mathsf{s}_1^+ + \mathsf{s}_2^+ \\
(\mathsf{s}_1 \to \mathsf{s}_2)^+ & = & \mathsf{s}_1^+ \to \mathsf{s}_2^+ \\
(\mathsf{T}_\ell\, \mathsf{s})^+ & = & \forall \boldsymbol{\beta}{::}*.\, ((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})) \to \boldsymbol{\beta}
\end{array}
$$

$\boxed{\Gamma^+}$

$$
\begin{array}{rcl}
(\cdot)^+ & = & \cdot \\
(\Gamma, \mathsf{x} : \mathsf{s})^+ & = & \Gamma^+, \mathbf{x} : \mathsf{s}^+
\end{array}
$$

Figure 11: DCC to F_ω: Lattice, Type, and Context Translation

We will often simply write $(\mathsf{T}_\ell \mathbf{t})^+$ to mean $\forall \boldsymbol{\beta}{::}*.\, ((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathbf{t} \to \boldsymbol{\beta})) \to \boldsymbol{\beta}$. Of course, $(\mathsf{T}_\ell \mathbf{t})^+$ doesn't make sense since $\mathbf{t}$ is already a target type, so read this as a macro for $\forall \boldsymbol{\beta}{::}*.\, ((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathbf{t} \to \boldsymbol{\beta})) \to \boldsymbol{\beta}$.

$\boxed{\preceq^{+}}$ proof constructors

$$
\begin{array}{lcl}
\mathbf{p_1} & : & \forall \boldsymbol{\beta}_\ell{::}{*}.\,(\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ \mathbf{1}),\\
\mathbf{p}_\times & : & \forall \boldsymbol{\beta}_\ell{::}{*}.\,\forall \boldsymbol{\alpha}_\mathbf{1}{::}{*}.\,\forall \boldsymbol{\alpha}_\mathbf{2}{::}{*}.\\
 & & \quad ((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\alpha}_\mathbf{1}) \times (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\alpha}_\mathbf{2})) \rightarrow (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ (\boldsymbol{\alpha}_\mathbf{1} \times \boldsymbol{\alpha}_\mathbf{2})),\\
\mathbf{p}_\rightarrow & : & \forall \boldsymbol{\beta}_\ell{::}{*}.\,\forall \boldsymbol{\alpha}_\mathbf{1}{::}{*}.\,\forall \boldsymbol{\alpha}_\mathbf{2}{::}{*}.\\
 & & \quad (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\alpha}_\mathbf{2}) \rightarrow (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ (\boldsymbol{\alpha}_\mathbf{1} \rightarrow \boldsymbol{\alpha}_\mathbf{2})),\\
\mathbf{p}_{\mathbf{T}_1} & : & \forall \boldsymbol{\beta}_\ell{::}{*}.\,\forall \boldsymbol{\beta}_{\ell'}{::}{*}.\,\forall \boldsymbol{\alpha}{::}{*}.\\
 & & \quad (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\alpha}) \rightarrow\\
 & & \qquad (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ (\forall \boldsymbol{\beta}{::}{*}.\,((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_{\ell'}\ \ \boldsymbol{\beta}) \times (\boldsymbol{\alpha} \rightarrow \boldsymbol{\beta})) \rightarrow \boldsymbol{\beta}))\\
\mathbf{p}_{\mathbf{T}_2} & : & \forall \boldsymbol{\beta}_\ell{::}{*}.\,\forall \boldsymbol{\beta}_{\ell'}{::}{*}.\,\forall \boldsymbol{\alpha}{::}{*}.\\
 & & \quad (\boldsymbol{\beta}_{\ell'} \rightarrow \boldsymbol{\beta}_\ell) \rightarrow\\
 & & \qquad (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_\ell\ \ (\forall \boldsymbol{\beta}{::}{*}.\,((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\beta}_{\ell'}\ \ \boldsymbol{\beta}) \times (\boldsymbol{\alpha} \rightarrow \boldsymbol{\beta})) \rightarrow \boldsymbol{\beta})),
\end{array}
$$

$\boxed{\mathbf{pf}[\![\ell \preceq \mathsf{s}]\!] : (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \mathsf{s}^{+})}$ proof-term construction

$$
\begin{array}{rcll}
\mathbf{pf}[\![\ell \preceq \mathbf{1}]\!] & \stackrel{\text{def}}{=} & \mathbf{p_1}\,[\boldsymbol{\alpha}_\ell] & \\
\mathbf{pf}[\![\ell \preceq \mathsf{s}_1 \times \mathsf{s}_2]\!] & \stackrel{\text{def}}{=} & \mathbf{p}_\times\,[\boldsymbol{\alpha}_\ell]\,[\mathsf{s}_1^{+}]\,[\mathsf{s}_2^{+}]\,\langle \mathbf{pf}[\![\ell \preceq \mathsf{s}_1]\!], \mathbf{pf}[\![\ell \preceq \mathsf{s}_2]\!]\rangle & \\
\mathbf{pf}[\![\ell \preceq \mathsf{s}_1 \rightarrow \mathsf{s}_2]\!] & \stackrel{\text{def}}{=} & \mathbf{p}_\rightarrow\,[\boldsymbol{\alpha}_\ell]\,[\mathsf{s}_1^{+}]\,[\mathsf{s}_2^{+}]\,\mathbf{pf}[\![\ell \preceq \mathsf{s}_2]\!] & \\
\mathbf{pf}[\![\ell \preceq \mathsf{T}_{\ell'}\ \mathsf{s}]\!] & \stackrel{\text{def}}{=} & \mathbf{p}_{\mathbf{T}_1}\,[\boldsymbol{\alpha}_\ell]\,[\boldsymbol{\alpha}_{\ell'}]\,[\mathsf{s}^{+}]\,\mathbf{pf}[\![\ell \preceq \mathsf{s}]\!] & \text{if } \ell \preceq \mathsf{s} \text{ and } \ell \not\sqsubseteq \ell'\\
 & & \mathbf{p}_{\mathbf{T}_2}\,[\boldsymbol{\alpha}_\ell]\,[\boldsymbol{\alpha}_{\ell'}]\,[\mathsf{s}^{+}]\,\mathbf{c}_{\ell'\ell} & \text{if } \ell \sqsubseteq \ell'
\end{array}
$$

Figure 12: F_ω: Protection Proofs

We use the name $\boldsymbol{\beta}_\ell$ to allude to the globally quantified $\boldsymbol{\alpha}_\ell \mid \ell \in \mathcal{L}_\ell$, as this type should always be instantiated with some $\boldsymbol{\alpha}_\ell$. We will enforce this property via parametricity later.

$\boxed{\Gamma \vdash e : s \rightsquigarrow \mathbf{m}}$ where $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : s^+$

$$\text{(DF-Unit)}\ \frac{}{\Gamma \vdash \langle\rangle : 1 \rightsquigarrow \langle\rangle} \qquad \text{(DF-Pair)}\ \frac{\Gamma \vdash e_1 : s_1 \rightsquigarrow \mathbf{m_1} \qquad \Gamma \vdash e_2 : s_2 \rightsquigarrow \mathbf{m_2}}{\Gamma \vdash \langle e_1, e_2\rangle : s_1 \times s_2 \rightsquigarrow \langle \mathbf{m_1}, \mathbf{m_2}\rangle} \qquad \text{(DF-Prj)}\ \frac{\Gamma \vdash e : s_1 \times s_2 \rightsquigarrow \mathbf{m}}{\Gamma \vdash \mathsf{prj_i}\ e : s_i \rightsquigarrow \mathbf{prj_i\ m}}$$

$$\text{(DF-Sum)}\ \frac{\Gamma \vdash e : s_i \rightsquigarrow \mathbf{m}}{\Gamma \vdash \mathsf{inj_i}\ e : s_1 + s_2 \rightsquigarrow \mathbf{inj_i\ m}}$$

$$\text{(DF-Case)}\ \frac{\Gamma \vdash e : s_1 + s_2 \rightsquigarrow \mathbf{m} \qquad \Gamma, x : s_1 \vdash e_1 : s \rightsquigarrow \mathbf{m_1} \qquad \Gamma, x : s_2 \vdash e_2 : s \rightsquigarrow \mathbf{m_2}}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ \mathsf{inj_1}\ x.\ e_1 \parallel \mathsf{inj_2}\ x.\ e_2 : s \rightsquigarrow \mathbf{case\ m\ of\ inj_1\ x.\ m_1} \parallel \mathbf{inj_2\ x.\ m_2}}$$

$$\text{(DF-Var)}\ \frac{(x : s)\ \in \Gamma}{\Gamma \vdash x : s \rightsquigarrow \mathbf{x}} \qquad \text{(DF-Fun)}\ \frac{\Gamma, x : s \vdash e : s_2 \rightsquigarrow \mathbf{m}}{\Gamma \vdash \lambda x : s_1.\ e : s_1 \to s_2 \rightsquigarrow \boldsymbol{\lambda}\mathbf{x}{:}s_1^+.\mathbf{m}}$$

$$\text{(DF-App)}\ \frac{\Gamma \vdash e_1 : s_1 \to s_2 \rightsquigarrow \mathbf{m_1} \qquad \Gamma \vdash e_2 : s_1 \rightsquigarrow \mathbf{m_2}}{\Gamma \vdash e_1\ e_2 : s_2 \rightsquigarrow \mathbf{m_1\ m_2}}$$

$$\text{(DF-Prot)}\ \frac{\Gamma \vdash e : s \rightsquigarrow \mathbf{m}}{\Gamma \vdash \eta_\ell\ e : \mathsf{T}_\ell\ s \rightsquigarrow \boldsymbol{\Lambda\beta}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (s^+ \to \boldsymbol{\beta})).((\mathbf{prj_2\ x})\ \ \mathbf{m})}$$

$$\text{(DF-Bind)}\ \frac{\Gamma \vdash e_1 : \mathsf{T}_\ell\ s_1 \rightsquigarrow \mathbf{m_1} \qquad \Gamma, x : s_1 \vdash e_2 : s_2 \rightsquigarrow \mathbf{m_2} \qquad \ell \preceq s_2}{\Gamma \vdash \mathsf{bind}\ x = e_1\ \mathsf{in}\ e_2 : s_2 \rightsquigarrow \mathbf{m_1}\ [(s_2)^+]\ \langle \mathbf{pf}[\![\ell \preceq s_2]\!], (\boldsymbol{\lambda}\mathbf{x}{:}s_1^+.\mathbf{m_2})\rangle}$$

Figure 13: DCC to F_ω: Term Translation

3.1 Definition of open contexts

$$
\begin{array}{rcl}
\mathbf{D}_\ell & = & \{\hat{\boldsymbol{\alpha}}_\ell :: * \mid \ell \in \mathcal{L}_\ell\} \cup \{\hat{\boldsymbol{\alpha}}_\preceq :: * \to * \to *\} \\
\mathbf{G}_\ell & = & \{\hat{\mathbf{x}}_\ell : \hat{\boldsymbol{\alpha}}_\ell \mid \ell \in \mathcal{L}_\ell\} \\
\mathbf{G}_\preceq & = & \{\hat{\mathbf{p}}_\mathbf{1} : \forall \boldsymbol{\beta}_\ell :: *.\, (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ \mathbf{1}), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \\
 & & \quad \hat{\mathbf{p}}_\times : \forall \boldsymbol{\beta}_\ell :: *.\, \forall \boldsymbol{\alpha}_\mathbf{1} :: *.\, \forall \boldsymbol{\alpha}_\mathbf{2} :: *.\, ((\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ \boldsymbol{\alpha}_\mathbf{1}) \times (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ \boldsymbol{\alpha}_\mathbf{2})) \to (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ (\boldsymbol{\alpha}_\mathbf{1} \times \boldsymbol{\alpha}_\mathbf{2})), \\
 & & \quad \hat{\mathbf{p}}_\to : \forall \boldsymbol{\beta}_\ell :: *.\, \forall \boldsymbol{\alpha}_\mathbf{1} :: *.\, \forall \boldsymbol{\alpha}_\mathbf{2} :: *.\, (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ \boldsymbol{\alpha}_\mathbf{2}) \to (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ (\boldsymbol{\alpha}_\mathbf{1} \to \boldsymbol{\alpha}_\mathbf{2})), \\
 & & \quad \hat{\mathbf{p}}_{\mathbf{T}_1} : \forall \boldsymbol{\beta}_\ell :: *.\, \forall \boldsymbol{\beta}_{\ell'} :: *.\, (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_{\ell'} \ \mathbf{t}) \to (\hat{\boldsymbol{\alpha}}_\preceq \ \hat{\boldsymbol{\alpha}}_\ell \ (\forall \boldsymbol{\beta} :: *.\, ((\boldsymbol{\alpha}_\preceq \ \boldsymbol{\beta}_{\ell'} \ \boldsymbol{\beta}) \times (\mathbf{t} \to \boldsymbol{\beta})) \to \boldsymbol{\beta})) \\
 & & \quad \hat{\mathbf{p}}_{\mathbf{T}_2} : \forall \boldsymbol{\beta}_\ell :: *.\, \forall \boldsymbol{\beta}_{\ell'} :: *.\, \forall \boldsymbol{\alpha} :: *.\, (\boldsymbol{\beta}_{\ell'} \to \boldsymbol{\beta}_\ell) \to (\hat{\boldsymbol{\alpha}}_\preceq \ \boldsymbol{\beta}_\ell \ (\forall \boldsymbol{\beta} :: *.\, ((\boldsymbol{\alpha}_\preceq \ \boldsymbol{\beta}_{\ell'} \ \boldsymbol{\beta}) \times (\boldsymbol{\alpha} \to \boldsymbol{\beta})) \to \boldsymbol{\beta})), \\
\boldsymbol{\Sigma} & = & \mathbf{D}_\ell ; \mathbf{G}_\ell, \mathbf{G}_\preceq \\
\boldsymbol{\Sigma}_\mathbf{D} & = & \mathbf{D}_\ell \\
\boldsymbol{\Sigma}_\mathbf{G} & = & \mathbf{G}_\ell, \mathbf{G}_\preceq
\end{array}
$$

Figure 14: Definition of open context

$$[\![\mathcal{L}_{\sqsubseteq}^{+}]\!] \quad = \quad \{\, \mathbf{c}_{\ell\ell'} \mapsto \boldsymbol{\lambda}\mathbf{x}{:}\hat{\boldsymbol{\alpha}}_{\ell}.\hat{\mathbf{x}}_{\ell'} \ \mid\ \ell' \sqsubseteq \ell \in \mathcal{L}_{\sqsubseteq} \,\}$$

Figure 15: Interpretation of Coercions

$$[\![\underline{\preceq}^{+}]\!] \quad = \quad \left\{ \begin{array}{l} \mathbf{p_1} \mapsto \boldsymbol{\Lambda\beta}_{\ell}{::}{*}.\hat{\mathbf{p}}_{\mathbf{1}}\,[\boldsymbol{\beta}_{\ell}]\,[\mathbf{1}], \\ \mathbf{p}_{\times} \mapsto \boldsymbol{\Lambda\beta}_{\ell}{::}{*}.\boldsymbol{\Lambda\alpha_1}{::}{*}.\boldsymbol{\Lambda\alpha_2}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_{\underline{\preceq}}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha_1}) \times (\hat{\boldsymbol{\alpha}}_{\underline{\preceq}}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha_2})).\hat{\mathbf{p}}_{\times}\,[\boldsymbol{\beta}_{\ell}]\,[\boldsymbol{\alpha_1}]\,[\boldsymbol{\alpha_2}]\,\mathbf{x}, \\ \mathbf{p}_{\rightarrow} \mapsto \boldsymbol{\Lambda\beta}_{\ell}{::}{*}.\boldsymbol{\Lambda\alpha_1}{::}{*}.\boldsymbol{\Lambda\alpha_2}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}(\hat{\boldsymbol{\alpha}}_{\underline{\preceq}}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha_2}).\hat{\mathbf{p}}_{\rightarrow}\,[\boldsymbol{\beta}_{\ell}]\,[\boldsymbol{\alpha_1}]\,[\boldsymbol{\alpha_2}]\,\mathbf{x}, \\ \mathbf{p}_{\mathbf{T}_1} \mapsto \boldsymbol{\Lambda\beta}_{\ell}{::}{*}.\boldsymbol{\Lambda\beta}_{\ell'}{::}{*}.\boldsymbol{\Lambda\alpha}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}(\hat{\boldsymbol{\alpha}}_{\underline{\preceq}}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha}).\hat{\mathbf{p}}_{\mathbf{T}_1}\,[\boldsymbol{\beta}_{\ell}]\,[\boldsymbol{\beta}_{\ell'}]\,\mathbf{x} \\ \mathbf{p}_{\mathbf{T}_2} \mapsto \boldsymbol{\Lambda\beta}_{\ell}{::}{*}.\boldsymbol{\Lambda\beta}_{\ell'}{::}{*}.\boldsymbol{\Lambda\alpha}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}(\boldsymbol{\beta}_{\ell'} \rightarrow \boldsymbol{\beta}_{\ell}).\hat{\mathbf{p}}_{\mathbf{T}_2}\,[\boldsymbol{\beta}_{\ell}]\,[\boldsymbol{\beta}_{\ell'}]\,[\boldsymbol{\alpha}]\,\mathbf{x}, \end{array} \right\}$$

Figure 16: Interpretation of Proof Constructors

3.2 Observer-sensitive F_ω relation

$$
\begin{array}{lcl}
[\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_\zeta & = & \{\ \boldsymbol{\alpha}_\ell :: * \mapsto (\hat{\boldsymbol{\alpha}}_\ell, \hat{\boldsymbol{\alpha}}_\ell, \mathit{Atom}\,[\hat{\boldsymbol{\alpha}}_\ell, \hat{\boldsymbol{\alpha}}_\ell]^{\boldsymbol{\Sigma}}) \quad \} \\
& \cup & \\
& & \left\{ \begin{array}{l}
\boldsymbol{\alpha}_\preceq :: * \to * \to * \mapsto \\
\quad (\boldsymbol{\lambda\beta}_\ell{::}*.\boldsymbol{\lambda\beta}{::}*.(\hat{\boldsymbol{\alpha}}_\preceq\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\beta}),\ \boldsymbol{\lambda\beta}_\ell{::}*.\boldsymbol{\lambda\beta}{::}*.(\hat{\boldsymbol{\alpha}}_\preceq\ \ \boldsymbol{\beta}_\ell\ \ \boldsymbol{\beta}), \\
\quad \lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_\ell).\lambda_R(\mathbf{t}'_1, \mathbf{t}'_2, \mathbf{R}_{\boldsymbol{\beta}}). \\
\qquad \{(\mathbf{m_1}, \mathbf{m_2}) \in \mathit{Atom}\,[(\hat{\boldsymbol{\alpha}}_\preceq\ \ \mathbf{t}_1\ \ \mathbf{t}'_1), (\hat{\boldsymbol{\alpha}}_\preceq\ \ \mathbf{t}_2\ \ \mathbf{t}'_2)]^{\boldsymbol{\Sigma}} \mid \mathbf{t}_1 = \hat{\boldsymbol{\alpha}}_\ell \wedge \mathbf{t}_2 = \hat{\boldsymbol{\alpha}}_\ell \wedge \\
\qquad \exists \mathsf{s}^+_1.\mathbf{t}'_1 = \mathsf{s}^+_1 \wedge \ell \preceq \mathsf{s}_1\ \wedge \\
\qquad \exists \mathsf{s}^+_2.\mathbf{t}'_2 = \mathsf{s}^+_2 \wedge \ell \preceq \mathsf{s}_2\ \wedge \\
\qquad (\ell \not\sqsubseteq \zeta\ \implies\ \mathbf{R}_{\boldsymbol{\beta}} = \mathit{Atom}\,[\mathbf{t}'_1, \mathbf{t}'_2]^{\boldsymbol{\Sigma}})\})
\end{array} \right\}
\end{array}
$$

Figure 17: F_ω: Relation Interpretation of Lattice and $\ell \preceq \mathsf{s}$

$$
\begin{array}{lcl}
\mathcal{L}^+_\ell; \mathcal{L}^+_\sqsubseteq, \preceq^+, \Gamma^+ \vdash \mathbf{m_1} \approx_\zeta \mathbf{m_2} : \mathsf{s}^+ & \stackrel{\text{def}}{=} & \mathcal{L}^+_\ell; \mathcal{L}^+_\sqsubseteq, \preceq^+, \Gamma^+ \vdash \mathbf{m_1} : \mathsf{s}^+\ \wedge\ \mathcal{L}^+_\ell; \mathcal{L}^+_\sqsubseteq, \preceq^+, \Gamma^+ \vdash \mathbf{m_2} : \mathsf{s}^+\ \wedge \\
& & \forall \boldsymbol{\rho}, \boldsymbol{\gamma}_\sqsubseteq, \boldsymbol{\gamma}_\preceq, \boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}.\ \boldsymbol{\rho} = [\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_\zeta \wedge\ \boldsymbol{\gamma}_\sqsubseteq = [\![\mathcal{L}^+_\sqsubseteq]\!] \wedge\ \boldsymbol{\gamma}_\preceq = [\![\preceq^+]\!]\ \wedge \\
& & \quad (\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}) \in \boldsymbol{\mathcal{G}}[\![\Gamma^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}\ \implies \\
& & \qquad (\boldsymbol{\rho_1}\boldsymbol{\gamma}_\sqsubseteq\boldsymbol{\gamma}_\preceq\boldsymbol{\gamma_1}(\mathbf{m_1}), \boldsymbol{\rho_2}\boldsymbol{\gamma}_\sqsubseteq\boldsymbol{\gamma}_\preceq\boldsymbol{\gamma_2}(\mathbf{m_2})) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}
\end{array}
$$

Figure 18: F_ω: Observer-sensitive Logical Relation

3.3 DCC to F_ω Logical Relation

In this subsection we define an open, cross-language logical relation, relating DCC to F_ω.

$$
\begin{array}{rcl}
\eta_{\mathbf{k}}^{\ell,\mathsf{s}} & \stackrel{\text{def}}{=} & \boldsymbol{\lambda}\mathbf{y}{:}\mathsf{s}^{+}.\boldsymbol{\Lambda\beta}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_{\ell}\ \ \boldsymbol{\beta}) \times (\mathsf{s}^{+} \to \boldsymbol{\beta})).((\mathbf{prj}_2\ \mathbf{x})\ \ \mathbf{y}) \\[1em]
\mathit{Atom}^{+}[\mathsf{s}]_{\boldsymbol{\delta}} & = & \{\,(\mathsf{e},\mathbf{m}) \mid\ \cdot \vdash \mathsf{e} : \mathsf{s}\ \wedge\ \boldsymbol{\Sigma}_{\mathbf{D}} \vdash \boldsymbol{\delta}(\mathsf{s}^{+})\ \wedge\ \boldsymbol{\Sigma}_{\mathbf{D}};\boldsymbol{\Sigma}_{\mathbf{G}} \vdash \mathbf{m} : \boldsymbol{\delta}(\mathsf{s}^{+})\,\} \\
\mathit{Atom}^{+\mathrm{val}}[\mathsf{s}]_{\boldsymbol{\delta}} & = & \{\,(\mathsf{v},\mathbf{u}) \mid\ \cdot \vdash \mathsf{v} : \mathsf{s}\ \wedge\ \boldsymbol{\Sigma}_{\mathbf{D}} \vdash \boldsymbol{\delta}(\mathsf{s}^{+})\ \wedge\ \boldsymbol{\Sigma}_{\mathbf{D}};\boldsymbol{\Sigma}_{\mathbf{G}} \vdash \mathbf{u} : \boldsymbol{\delta}(\mathsf{s}^{+})\,\} \\[0.5em]
\mathcal{V}_{\zeta}^{+}[\![\mathsf{1}]\!]_{\boldsymbol{\delta}} & = & \{\,(\langle\rangle,\langle\rangle) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{1}]_{\boldsymbol{\delta}}\,\} \\
\mathcal{V}_{\zeta}^{+}[\![\mathsf{s}\times\mathsf{s}']\!]_{\boldsymbol{\delta}} & = & \{(\langle\mathsf{e},\mathsf{e}'\rangle,\langle\mathbf{m},\mathbf{m}'\rangle) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{s}\times\mathsf{s}']_{\boldsymbol{\delta}} \mid \\
 & & \qquad (\mathsf{e},\mathbf{m}) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}\ \wedge\ (\mathsf{e}',\mathbf{m}') \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}\} \\
\mathcal{V}_{\zeta}^{+}[\![\mathsf{s}+\mathsf{s}']\!]_{\boldsymbol{\delta}} & = & \{(\mathsf{inj}_1\,\mathsf{e},\mathbf{inj}_1\ \mathbf{m}) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{s}+\mathsf{s}']_{\boldsymbol{\delta}} \mid\ (\mathsf{e},\mathbf{m}) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}\} \\
 & \cup & \{(\mathsf{inj}_2\,\mathsf{e},\mathbf{inj}_2\ \mathbf{m}) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{s}+\mathsf{s}']_{\boldsymbol{\delta}} \mid\ (\mathsf{e},\mathbf{m}) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}\} \\
\mathcal{V}_{\zeta}^{+}[\![\mathsf{s}'\to\mathsf{s}]\!]_{\boldsymbol{\delta}} & = & \{(\lambda\mathsf{x}:\mathsf{s}'.\,\mathsf{e},\boldsymbol{\lambda}\mathbf{x}{:}\boldsymbol{\delta}((\mathsf{s}')^{+}).\mathbf{m}) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{s}'\to\mathsf{s}]_{\boldsymbol{\delta}} \mid \\
 & & \qquad \forall\mathsf{e}',\mathbf{m}'.(\mathsf{e}',\mathbf{m}') \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}']\!]_{\boldsymbol{\delta}} \implies (\mathsf{e}[\mathsf{e}'/\mathsf{x}],\mathbf{m}[\mathbf{m}'/\mathbf{x}]) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}\} \\
\mathcal{V}_{\zeta}^{+}[\![\mathsf{T}_{\ell}\ \mathsf{s}]\!]_{\boldsymbol{\delta}} & = & \{(\eta_{\ell}\,\mathsf{e},\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m}) \in \mathit{Atom}^{+\mathrm{val}}[\mathsf{T}_{\ell}\ \mathsf{s}]_{\boldsymbol{\delta}} \mid \\
 & & \qquad \text{Let } \boldsymbol{\rho} = [\![\mathcal{L}_{\ell}^{+}]\!]_{\zeta}^{\boldsymbol{\Sigma}}. \\
 & & \qquad \exists\mathbf{m}'.\boldsymbol{\Sigma}_{\mathbf{D}};\boldsymbol{\Sigma}_{\mathbf{G}} \vdash \mathbf{m}' : \boldsymbol{\delta}(\mathsf{s}^{+})\ \wedge \\
 & & \qquad (\mathbf{m}[(\mathsf{T}_{\ell}\ \mathsf{s})^{+}/\boldsymbol{\beta}]\ \langle\mathbf{pf}[\![\ell \preceq \mathsf{T}_{\ell}\ \mathsf{s}]\!],\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle,\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}') \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_{\ell}\ \mathsf{s})^{+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}\ \wedge \\
 & & \qquad (\mathsf{e},\mathbf{m}') \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}^{+}]\!]_{\boldsymbol{\delta}}\} \\[0.5em]
\mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}} & = & \{(\mathsf{e},\mathbf{m}) \in \mathit{Atom}^{+}[\mathsf{s}]_{\boldsymbol{\delta}} \mid \\
 & & \exists\mathsf{v},\mathbf{u}.\mathsf{e} \longmapsto^{*} \mathsf{v}\ \wedge\ \mathbf{m} \longmapsto^{*} \mathbf{u}\ \wedge\ (\mathsf{v},\mathbf{u}) \in \mathcal{V}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}\} \\[0.5em]
\mathcal{G}_{\zeta}^{+}[\![\cdot]\!]_{\boldsymbol{\delta}} & = & \{\,(\emptyset,\emptyset)\,\} \\
\mathcal{G}_{\zeta}^{+}[\![\Gamma,\mathsf{x}:\mathsf{s}]\!]_{\boldsymbol{\delta}} & = & \{\,(\gamma\,[\mathsf{x}\mapsto\mathsf{e}]\,,\boldsymbol{\gamma}\,[\mathbf{x}\mapsto\mathbf{m}]) \mid (\gamma,\boldsymbol{\gamma}) \in \mathcal{G}_{\zeta}^{+}[\![\Gamma]\!]_{\boldsymbol{\delta}} \wedge (\mathsf{e},\mathbf{m}) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}\,\} \\[0.5em]
\Gamma \mid \boldsymbol{\Sigma} \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s} \mid \boldsymbol{\delta} & \stackrel{\text{def}}{=} & \Gamma \vdash \mathsf{e} : \mathsf{s}\ \wedge\ \boldsymbol{\Sigma}_{\mathbf{D}};\boldsymbol{\Sigma}_{\mathbf{G}},\hat{\Gamma^{+}} \vdash \mathbf{m} : \boldsymbol{\delta}(\mathsf{s}^{+})\ \wedge \\
 & & \forall\zeta,\gamma,\boldsymbol{\gamma}.\ (\gamma,\boldsymbol{\gamma}) \in \mathcal{G}_{\zeta}^{+}[\![\Gamma]\!]_{\boldsymbol{\delta}}.\ \implies \\
 & & \qquad (\gamma(\mathsf{e}),\boldsymbol{\delta}(\boldsymbol{\gamma}(\mathbf{m}))) \in \mathcal{E}_{\zeta}^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}) \\[0.5em]
\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s} & \stackrel{\text{def}}{=} & \Gamma \vdash \mathsf{e} : \mathsf{s}\ \wedge\ \mathcal{L}_{\ell}^{+};\mathcal{L}_{\sqsubseteq}^{+},\preceq^{+},\Gamma^{+} \vdash \mathbf{m} : \mathsf{s}^{+}\ \wedge \\
 & & \qquad (\forall\boldsymbol{\delta},\boldsymbol{\gamma}_{\sqsubseteq},\boldsymbol{\gamma}_{\preceq}. \qquad\qquad\qquad\qquad\qquad\qquad) \\
 & & \qquad \boldsymbol{\delta} = \{\boldsymbol{\alpha}_{\ell} \mapsto \hat{\boldsymbol{\alpha}}_{\ell}\ \mid\ \ell \in \mathcal{L}_{\ell}\} \cup \{\boldsymbol{\alpha}_{\preceq} \mapsto \hat{\boldsymbol{\alpha}}_{\preceq}\}\ \wedge \\
 & & \qquad \boldsymbol{\gamma}_{\sqsubseteq} = [\![\mathcal{L}_{\sqsubseteq}^{+}]\!]\ \wedge\ \boldsymbol{\gamma}_{\preceq} = [\![\preceq^{+}]\!]\ \implies \\
 & & \qquad \Gamma \mid \boldsymbol{\Sigma} \vdash \mathsf{e} \simeq \boldsymbol{\delta}(\boldsymbol{\gamma}_{\sqsubseteq}(\boldsymbol{\gamma}_{\preceq}(\mathbf{m}))) : \mathsf{s} \mid \boldsymbol{\delta}
\end{array}
$$

Figure 19: Logical Relation between DCC and F_ω

4 Back-translation

In this section, we present a back-translation relation between target terms of translation type and source terms. We need this back-translation when proving that the translation is equivalence preserving.

4.1 Context typing

$\boxed{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E} : \mathbf{t_1}\Rightarrow\mathbf{t_2}}$

$$\text{(FTE-Hole)}\ \frac{}{\mathbf{\Delta};\mathbf{\Gamma} \vdash [\cdot]_{\mathbf{T}} : \mathbf{t}\Rightarrow\mathbf{t}} \qquad \text{(FTE-Prj)}\ \frac{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E} : \mathbf{t}\Rightarrow(\mathbf{t_1} \times \mathbf{t_2})}{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{prj_i}\ \mathbf{E} : \mathbf{t}\Rightarrow\mathbf{t}_i}$$

$$\text{(FTE-Case)}\ \frac{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E} : \mathbf{t}\Rightarrow(\mathbf{t_1} + \mathbf{t_2}) \qquad \mathbf{\Delta};\mathbf{\Gamma},\mathbf{x_1} : \mathbf{t_1} \vdash \mathbf{m_1} : \mathbf{t'} \qquad \mathbf{\Delta};\mathbf{\Gamma},\mathbf{x_2} : \mathbf{t_2} \vdash \mathbf{m_2} : \mathbf{t'}}{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{case}\ \mathbf{E}\ \mathbf{of}\ \mathbf{inj_1}\ \mathbf{x_1}.\ \mathbf{m_1} \parallel \mathbf{inj_2}\ \mathbf{x_2}.\ \mathbf{m_2} : \mathbf{t}\Rightarrow\mathbf{t'}}$$

$$\text{(FTE-App)}\ \frac{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E} : \mathbf{t}\Rightarrow(\mathbf{t_1} \to \mathbf{t_2}) \qquad \mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t_1}}{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E}\ \mathbf{m_1} : \mathbf{t}\Rightarrow\mathbf{t_2}} \qquad \text{(FTE-Inst)}\ \frac{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E} : \mathbf{t_1}\Rightarrow(\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t_2}) \qquad \mathbf{\Delta} \vdash \mathbf{t'} :: \boldsymbol{\kappa}}{\mathbf{\Delta};\mathbf{\Gamma} \vdash \mathbf{E}\ [\mathbf{t'}] : \mathbf{t_1}\Rightarrow\mathbf{t_2}}$$

Figure 20: F_ω: Context Typing

$\boxed{\Gamma \vdash \mathsf{E} : \mathsf{s_1}\Rightarrow\mathsf{s_2}}$

$$\text{(DTE-Hole)}\ \frac{}{\Gamma \vdash [\cdot]_{\mathsf{S}} : \mathsf{s}\Rightarrow\mathsf{s}} \qquad \text{(DTE-Prj)}\ \frac{\Gamma \vdash \mathsf{E} : \mathsf{s}\Rightarrow(\mathsf{s_1} \times \mathsf{s_2})}{\Gamma \vdash \mathsf{prj_i}\ \mathsf{E} : \mathsf{s}\Rightarrow\mathsf{s_i}}$$

$$\text{(DTE-Case)}\ \frac{\Gamma \vdash \mathsf{E} : \mathsf{s}\Rightarrow(\mathsf{s_1} + \mathsf{s_2}) \qquad \Gamma, \mathsf{x_1} : \mathsf{s_1} \vdash \mathsf{e_1} : \mathsf{s'} \qquad \Gamma, \mathsf{x_2} : \mathsf{s_2} \vdash \mathsf{e_2} : \mathsf{s'}}{\Gamma \vdash \mathsf{case}\ \mathsf{E}\ \mathsf{of}\ \mathsf{inj_1}\ \mathsf{x_1}.\,\mathsf{e_1} \parallel \mathsf{inj_2}\ \mathsf{x_2}.\,\mathsf{e_2} : \mathsf{s}\Rightarrow\mathsf{s'}}$$

$$\text{(DTE-App)}\ \frac{\Gamma \vdash \mathsf{E} : \mathsf{s}\Rightarrow(\mathsf{s_1} \to \mathsf{s_2}) \qquad \Gamma \vdash \mathsf{e_1} : \mathsf{s_1}}{\Gamma \vdash \mathsf{E}\ \mathsf{e_1} : \mathsf{s}\Rightarrow\mathsf{s_2}} \qquad \text{(DTE-Bind)}\ \frac{\Gamma \vdash \mathsf{E} : \mathsf{s_1}\Rightarrow(\mathsf{T}_\ell\ \mathsf{s}) \qquad \Gamma, \mathsf{x} : \mathsf{s} \vdash \mathsf{e} : \mathsf{s_2} \qquad \ell \preceq \mathsf{s_2}}{\Gamma \vdash \mathsf{bind}\ \mathsf{x} = \mathsf{E}\ \mathsf{in}\ \mathsf{e} : \mathsf{s_1}\Rightarrow\mathsf{s_2}}$$

Figure 21: DCC: Context Typing

4.2 Back-translation

$$(\mathsf{T}_\ell\,\hat{}\,\mathsf{s})^+ = \forall\boldsymbol{\beta}{::}{*}.\,((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\beta})) \to \boldsymbol{\beta}$$

Figure 22: Back-translation types

$$\textit{Pf Environment} \quad \mathbf{G_k} \quad ::= \quad \cdot \mid \mathbf{G_k}, \mathbf{k} : (\mathsf{s}^{\hat{+}} \rightarrow (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}})$$

Figure 23: F_ω: Back Translation Context Grammar

$$\boxed{\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{s}]\!] \stackrel{\text{def}}{=} \boldsymbol{\delta}(\boldsymbol{\gamma}_{\sqsubseteq}(\boldsymbol{\gamma}_{\preceq}(\mathbf{pf}[\![\ell \preceq \mathsf{s}]\!])))} \text{ where } \begin{aligned} \boldsymbol{\delta} &= \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_{\preceq} \mapsto \hat{\boldsymbol{\alpha}}_{\preceq}\} \\ \boldsymbol{\gamma}_{\sqsubseteq} &= [\![\mathcal{L}_{\sqsubseteq}^{+}]\!] \\ \boldsymbol{\gamma}_{\preceq} &= [\![\preceq^{+}]\!] \end{aligned}$$

Figure 24: Back-translation proof constructor

$$
\begin{array}{lll}
F_\omega \textit{ One-layer context} & \mathbf{F} ::= & \mathbf{prj_i}\ [\cdot]_{\mathbf{T}} \mid [\cdot]_{\mathbf{T}}\ \mathbf{m} \mid [\cdot]_{\mathbf{T}}\ [\mathbf{t}] \mid \\
& & \mathbf{case}\ [\cdot]_{\mathbf{T}}\ \mathbf{of\ inj_1\ y.\ m_1} \parallel \mathbf{inj_2\ y.\ m_2} \\
\textit{DCC One-layer context} & \mathsf{F} ::= & \mathsf{prj_i}\ [\cdot]_{\mathsf{S}} \mid [\cdot]_{\mathsf{S}}\ \mathsf{e} \mid \mathsf{bind}\ \mathsf{x} = [\cdot]_{\mathsf{S}}\ \mathsf{in}\ \mathsf{e} \mid \\
& & \mathsf{case}\ [\cdot]_{\mathsf{S}}\ \mathsf{of\ inj_1\ y.\ e_1} \parallel \mathsf{inj_2\ y.\ e_2}
\end{array}
$$

$$
\boxed{\mathbf{E}^{\not\uparrow} = \mathbf{F}_0[\mathbf{F}_1[...\mathbf{F}_n]]}\ \text{where } \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{F}_0 : \mathbf{t_1} \Rightarrow \mathsf{s}^{\hat{+}}
$$
$$
\forall i \in [0,n]. \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{F}_i : \mathbf{t_{i+1}} \Rightarrow \mathbf{t}_i
$$
$$
\forall i \in [0, n+1].\ \nexists \mathsf{s_i}.\mathbf{t}_i = \mathsf{s}_i^{\hat{+}}
$$

$$
\boxed{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}}\ \text{where } \Gamma^{\hat{+}} ::= \cdot \mid \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}}
$$
$$
\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}}
$$
$$
(\Gamma^{\hat{+}})^{\hat{\uparrow}} \vdash \mathsf{e} : \mathsf{s}
$$

$$
\boxed{(\Gamma^{\hat{+}})^{\hat{\uparrow}}}\quad
\begin{array}{lll}
(\cdot)^{\hat{\uparrow}} & = & \cdot \\
(\Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}})^{\hat{\uparrow}} & = & (\Gamma^{\hat{+}})^{\hat{\uparrow}}, \mathsf{x} : \mathsf{s}
\end{array}
$$

$$
\text{(FD-Unit)}\ \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \langle\rangle : \mathbf{1} \uparrow \langle\rangle
\qquad
\text{(FD-Sum)}\ \frac{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_i^{\hat{+}} \uparrow \mathsf{e}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{inj_i\ m} : \mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{inj_i\ e}}
$$

$$
\text{(FD-Pair)}\ \frac{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_1} : \mathsf{s}_1^{\hat{+}} \uparrow \mathsf{e_1} \qquad \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_2} : \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{e_2}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \langle \mathbf{m_1}, \mathbf{m_2} \rangle : \mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}} \uparrow \langle \mathsf{e_1}, \mathsf{e_2} \rangle}
\qquad
\text{(FD-Var)}\ \frac{(\mathbf{x} : \mathsf{s}^{\hat{+}}) \in \Gamma^{\hat{+}}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{x} : \mathsf{s}^{\hat{+}} \uparrow \mathsf{x}}
$$

$$
\text{(FD-Fun)}\ \frac{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{e}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{\lambda x{:}s_1^{\hat{+}}.m} : \mathsf{s}_1^{\hat{+}} \to \mathsf{s}_2^{\hat{+}} \uparrow \lambda \mathsf{x} : \mathsf{s_1}.\ \mathsf{e}}
$$

$$
\text{(FD-Return)}\ \frac{\mathbf{\Sigma}; \mathbf{G_k}, \mathbf{k} : (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}); \Gamma^{\hat{+}} \vdash \mathbf{m}[(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}/\beta]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{k} \rangle : \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{\Lambda\beta{::}{*}.m} : \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}}
$$

$$
\text{(FD-K)}\ \frac{\mathbf{k} : (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}) \in \mathbf{G_k}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{k} : \mathsf{s}^{\hat{+}} \to \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}} \uparrow \lambda \mathsf{x} : \mathsf{s}.\ \eta_\ell\ \mathsf{x}}
$$

$$
\text{(FD-Bind)}\ \frac{\begin{array}{c}\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_p} : (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \mathsf{s}^{\hat{+}}) \times (\mathsf{s}'^{\hat{+}} \to \mathsf{s}^{\hat{+}}) \\ \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m} : (\mathsf{T}_\ell\ \mathsf{s}')^{\hat{+}} \uparrow \mathsf{e} \qquad \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{prj_2\ m_p} : \mathsf{s}'^{\hat{+}} \to \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}'\end{array}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m}\ [\mathsf{s}^{\hat{+}}]\ \mathbf{m_p} : \mathsf{s}^{\hat{+}} \uparrow \mathsf{bind}\ \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e}'\ \mathsf{x}}
$$

$$
\text{(FD-Subterm)}\ \frac{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{F} : \mathsf{s}_1^{\hat{+}} \Rightarrow \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{F}' \qquad \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_1^{\hat{+}} \uparrow \mathsf{e}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{F}[\mathbf{m}] : \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{F}'[\mathsf{e}]}
$$

$$
\text{(FD-Val)}\ \frac{\mathbf{E}^{\not\uparrow}[\mathbf{u}] \longmapsto \mathbf{m_1} \qquad \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_1} : \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{e}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{u}] : \mathsf{s}_2^{\hat{+}} \uparrow \mathsf{e}}
$$

$$
\text{(FD-Stuck)}\ \frac{\begin{array}{c}\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{case\ m\ of\ inj_1\ x_1.\ m_1} \parallel \mathbf{inj_2\ x_2.\ m_2} : \mathbf{t}\ \text{where}\ \nexists \mathsf{s}'.\mathbf{t} = \mathsf{s}'^{\hat{+}} \\ \mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{case\ m\ of\ inj_1\ y_1.\ E^{\not\uparrow}[m_1[y_1/x_1]]} \parallel \mathbf{inj_2\ y_2.\ E^{\not\uparrow}[m_2[y_2/x_2]]} : \mathsf{s}^{\hat{+}} \uparrow \mathsf{e} \qquad (\textit{fresh}\ \mathbf{y}_1', \mathbf{y}_2')\end{array}}{\mathbf{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{case\ m\ of\ inj_1\ x_1.\ m_1} \parallel \mathbf{inj_2\ x_2.\ m_2}] : \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}}
$$

Figure 25: F_ω to DCC Back-translation

$$\boxed{\Sigma; G_k; \Gamma^{\hat{+}} \vdash F : s_1^{\hat{+}} \Rightarrow s_2^{\hat{+}} \Uparrow F'} \text{ where } \begin{array}{l} \Sigma_D; \Sigma_G, G_k, \Gamma^{\hat{+}} \vdash F : s_1^{\hat{+}} \Rightarrow s_2^{\hat{+}} \\ (\Gamma^{\hat{+}})^{\hat{\Uparrow}} \vdash F' : s_1 \Rightarrow s_2 \end{array}$$

$$\text{(FD-Prj)} \frac{}{\Sigma; G_k; \Gamma^{\hat{+}} \vdash \mathbf{prj_i}\, [\cdot]_T : (s_1^{\hat{+}} \times s_1^{\hat{+}}) \Rightarrow s_i^{\hat{+}} \Uparrow \mathsf{prj_i}\, [\cdot]_S} \qquad \text{(FD-App)} \frac{\Sigma; G_k; \Gamma^{\hat{+}} \vdash \mathbf{m_2} : s_1^{\hat{+}} \Uparrow e_2}{\Sigma; G_k; \Gamma^{\hat{+}} \vdash [\cdot]_T\, \mathbf{m_2} : (s_1^{\hat{+}} \to s_2^{\hat{+}}) \Rightarrow s^{\hat{+}} \Uparrow [\cdot]_S\, e_2}$$

$$\text{(FD-Case)} \frac{\Sigma; G_k; \Gamma^{\hat{+}}, \mathbf{x} : s_1^{\hat{+}} \vdash \mathbf{m_1} : s^{\hat{+}} \Uparrow e_1 \qquad \Sigma; G_k; \Gamma^{\hat{+}}, \mathbf{x} : s_2^{\hat{+}} \vdash \mathbf{m_2} : s^{\hat{+}} \Uparrow e_2}{\begin{array}{c}\Sigma; G_k; \Gamma^{\hat{+}} \vdash \mathbf{case}\, [\cdot]_T\, \mathbf{of}\, \mathbf{inj_1}\, \mathbf{x_1}.\, \mathbf{m_1} \parallel \mathbf{inj_2}\, \mathbf{x_2}.\, \mathbf{m_2} : (s_1^{\hat{+}} + s_2^{\hat{+}}) \Rightarrow s^{\hat{+}} \Uparrow \\ \mathsf{case}\, [\cdot]_S\, \mathsf{of}\, \mathsf{inj_1}\, x_1.\, e_1 \parallel \mathsf{inj_2}\, x_2.\, e_2\end{array}}$$

Figure 26: Back-translation of **F** contexts

$$
\begin{aligned}
\mathbf{W} \quad &::= \quad (\mathbf{G}'_{\mathbf{k}}; \Gamma'^{\hat{+}}) \\
\mathbf{W}_{\mathbf{k}} \quad &\stackrel{\text{def}}{=} \quad \mathbf{G}'_{\mathbf{k}} \mid \mathbf{W} = (\mathbf{G}'_{\mathbf{k}}; \Gamma'^{\hat{+}}) \\
\mathbf{W}_{\Gamma} \quad &\stackrel{\text{def}}{=} \quad \Gamma'^{\hat{+}} \mid \mathbf{W} = (\mathbf{G}'_{\mathbf{k}}; \Gamma'^{\hat{+}}) \\
\mathbf{W} \supseteq \mathbf{W}' \quad &\stackrel{\text{def}}{=} \quad \mathbf{W}_{\mathbf{k}} \supseteq \mathbf{W}'_{\mathbf{k}} \wedge \mathbf{W}_{\Gamma} \supseteq \mathbf{W}'_{\Gamma}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Atom}^{\uparrow}[\mathbf{t}]^{\mathbf{\Sigma}}_{\delta} \quad &= \quad \{\, ((\mathbf{G}_{\mathbf{k}}; \Gamma^{\hat{+}}), \mathbf{m}) \mid \mathbf{\Sigma}_{\mathbf{D}}; \mathbf{\Sigma}_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \delta(\mathbf{t}) \,\} \\
\mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{*} \quad &= \quad \{(\mathsf{s}^{\hat{+}}, \mathbf{R}) \mid \mathbf{R} \subseteq \mathit{Atom}^{\uparrow}\left[\mathsf{s}^{\hat{+}}\right]^{\mathbf{\Sigma}}_{\emptyset} \ \wedge\ \forall(\mathbf{W}, \mathbf{m}) \in \mathbf{R}.\ \forall \mathbf{W}' \supseteq \mathbf{W}. \\
&\qquad\quad (\mathbf{W}', \mathbf{m}) \in \mathbf{R} \wedge \exists \mathsf{e}. \mathbf{\Sigma}_{\mathbf{D}}; \mathbf{\Sigma}_{\mathbf{G}}, \mathbf{W}'_{\mathbf{k}}, \mathbf{W}'_{\Gamma} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \uparrow \mathsf{e}\} \\
&\cup \\
&\qquad \{(\mathbf{t}, \mathbf{R}) \mid \mathbf{R} \subseteq \mathit{Atom}^{\uparrow}[\mathbf{t}]^{\mathbf{\Sigma}}_{\emptyset} \ \wedge\ \nexists \mathsf{s}. \mathbf{t} = \mathsf{s}^{\hat{+}} \wedge \\
&\qquad\quad \forall\ (\mathbf{W}, \mathbf{m}) \in \mathbf{R},\ \mathbf{W}' \supseteq \mathbf{W}. (\mathbf{W}', \mathbf{m}) \in \mathbf{R}\ \wedge \\
&\qquad\quad \forall\ \mathbf{E}^{\not\vdash}, \mathbf{t}. \\
&\qquad\qquad (\mathbf{\Sigma}_{\mathbf{D}}; \mathbf{\Sigma}_{\mathbf{G}}, \mathbf{W}_{\mathbf{k}}, \mathbf{W}_{\Gamma} \vdash \mathbf{E}^{\not\vdash} : \mathbf{t} \Rightarrow \mathsf{s}'^{\hat{+}}\ \wedge \\
&\qquad\qquad (\forall\ \mathbf{W}', \mathbf{u}.\ (\mathbf{W}' \supseteq \mathbf{W}\ \wedge\ \mathbf{\Sigma}_{\mathbf{D}}; \mathbf{\Sigma}_{\mathbf{G}}, \mathbf{W}'_{\mathbf{k}}, \mathbf{W}'_{\Gamma} \vdash \mathbf{u} : \mathbf{t}) \implies \exists \mathsf{e}.\ \mathbf{\Sigma}; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{E}^{\not\vdash}[\mathbf{u}] : \mathsf{s}'^{\hat{+}} \uparrow \mathsf{e})) \implies \\
&\qquad\qquad\quad \exists \mathsf{e}'.\ \mathbf{\Sigma}; \mathbf{W}_{\mathbf{k}}; \mathbf{W}_{\Gamma} \vdash \mathbf{E}^{\not\vdash}[\mathbf{m}] : \mathsf{s}'^{\hat{+}} \uparrow \mathsf{e}'\} \\
\mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2} \quad &= \quad \{(\mathbf{t}_1, \mathbf{R}_1) \mid \\
&\qquad\quad \forall(\mathbf{t}_2, \mathbf{R}_2) \in \mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1}.((\mathbf{t}_1\ \mathbf{t}_2), \mathbf{R}_1\ (\mathbf{t}_2, \mathbf{R}_2)) \in \mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_2} \wedge \\
&\qquad\qquad \forall(\mathbf{t}'_2, \mathbf{R}'_2) \in \mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1}. \\
&\qquad\qquad\quad (\mathbf{t}_2, \mathbf{R}_2) \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1} (\mathbf{t}'_2, \mathbf{R}'_2) \implies (\mathbf{R}_1\ (\mathbf{t}_2, \mathbf{R}_2)) \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_2} (\mathbf{R}_1\ (\mathbf{t}'_2, \mathbf{R}'_2))\}
\end{aligned}
$$

$$
\begin{aligned}
\delta \equiv^{\mathbf{\Sigma}} \delta' \quad &\stackrel{\text{def}}{=} \quad \forall \boldsymbol{\alpha} :: \boldsymbol{\kappa} \in \mathrm{dom}(\delta). \delta(\boldsymbol{\alpha}) \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}} \delta'(\boldsymbol{\alpha}) \\
(\mathbf{t}_1, \mathbf{R}_1) \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}} (\mathbf{t}_2, \mathbf{R}_2) \quad &\stackrel{\text{def}}{=} \quad \mathbf{t}_1 \equiv \mathbf{t}_2\ \wedge\ \mathbf{R}_1 \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}} \mathbf{R}_2 \\
\mathbf{R}_1 \equiv^{\mathbf{\Sigma}}_{*} \mathbf{R}_2 \quad &\stackrel{\text{def}}{=} \quad \forall \mathbf{W}, \mathbf{m}_1.\ (\mathbf{W}, \mathbf{m}_1) \in \mathbf{R}_1 \iff (\mathbf{W}, \mathbf{m}_1) \in \mathbf{R}_2 \\
\mathbf{R}_1 \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2} \mathbf{R}_2 \quad &\stackrel{\text{def}}{=} \quad \forall(\mathbf{t}', \mathbf{R}') \in \mathit{Rel}^{\uparrow}\,{}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_1}.\ (\mathbf{R}_1\ (\mathbf{t}', \mathbf{R}')) \equiv^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}_2} (\mathbf{R}_2\ (\mathbf{t}', \mathbf{R}'))
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}^{\uparrow}[\![\mathbf{t} :: *]\!]^{\mathbf{\Sigma}}_{\delta} \quad &= \quad \mathcal{E}^{\uparrow}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta} \\
\mathcal{T}^{\uparrow}[\![\boldsymbol{\alpha} :: \boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2]\!]^{\mathbf{\Sigma}}_{\delta} \quad &= \quad \delta_{\mathbf{R}}(\boldsymbol{\alpha}) \\
\mathcal{T}^{\uparrow}[\![\lambda \boldsymbol{\alpha} :: \boldsymbol{\kappa}_1 . \mathbf{t} :: \boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2]\!]^{\mathbf{\Sigma}}_{\delta} \quad &= \quad \lambda_R(\mathbf{t}_1, \mathbf{R}_1).\{\mathcal{T}^{\uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa}_2]\!]^{\mathbf{\Sigma}}_{\delta[\boldsymbol{\alpha} :: \boldsymbol{\kappa}_1 \mapsto (\mathbf{t}_1, \mathbf{R}_1)]}\} \\
\mathcal{T}^{\uparrow}[\![\mathbf{t}_1\ \mathbf{t}_2 :: \boldsymbol{\kappa}_2]\!]^{\mathbf{\Sigma}}_{\delta} \quad &= \quad (\mathcal{T}^{\uparrow}[\![\mathbf{t}_1 :: \boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2]\!]^{\mathbf{\Sigma}}_{\delta} \\
&\qquad\quad (\delta(\mathbf{t}_2), \mathcal{T}^{\uparrow}[\![\mathbf{t}_2 :: \boldsymbol{\kappa}_1]\!]^{\mathbf{\Sigma}}_{\delta}))
\end{aligned}
$$

Figure 27: Back-translation: Logical Relations (Higher-Order)

$$
\begin{array}{rcl}
\mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\mathbf{t}]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, ((\mathbf{G_k}; \Gamma^{\hat{+}}), \mathbf{u}) \mid \boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{u} : \delta(\mathbf{t}) \,\} \\
\mathit{Atom}^{\hat{\uparrow}\mathrm{ctx}}\left[\mathbf{t}, \mathsf{s}^{\hat{+}}\right]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, ((\mathbf{G_k}; \Gamma^{\hat{+}}), \mathbf{E}^{\not{}}) \mid \boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{E}^{\not{}} : \delta(\mathbf{t}) \Rightarrow \mathsf{s}^{\hat{+}} \,\} \\
\mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, (\mathbf{W}, \mathbf{m}) \mid \exists \mathsf{e}.(\mathbf{W}, \mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}}\left[\mathsf{s}^{\hat{+}}\right]^{\boldsymbol{\Sigma}}_{\delta} \wedge \boldsymbol{\Sigma}; \mathbf{W_k}; \mathbf{W}_{\Gamma} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \hat{\uparrow}\ \mathsf{e} \,\} \\
\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} \\
\mathcal{V}^{\hat{\uparrow}}[\![\boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \delta_{\mathbf{R}}(\boldsymbol{\alpha}) \\
\mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t} \times \mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, (\mathbf{W}, \langle \mathbf{m_1}, \mathbf{m_2} \rangle) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\mathbf{t} \times \mathbf{t}']^{\boldsymbol{\Sigma}}_{\delta} \mid (\mathbf{W}, \mathbf{m_1}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} \wedge (\mathbf{W}, \mathbf{m_2}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta} \,\} \\
\mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t} + \mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, (\mathbf{W}, \mathbf{inj_1}\ \mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\mathbf{t} + \mathbf{t}']^{\boldsymbol{\Sigma}}_{\delta} \mid (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} \,\} \\
 & \cup & \{\, (\mathbf{W}, \mathbf{inj_2}\ \mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\mathbf{t} + \mathbf{t}']^{\boldsymbol{\Sigma}}_{\delta} \mid (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta} \,\} \\
\mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}' \to \mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{ (\mathbf{W}, \boldsymbol{\lambda} \mathbf{x}{:}\mathbf{t}'.\mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\mathbf{t}' \to \mathbf{t}]^{\boldsymbol{\Sigma}}_{\delta} \mid \\
 & & \quad \forall \mathbf{W}' \sqsupseteq \mathbf{W}, (\mathbf{W}', \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta}.(\mathbf{W}', \mathbf{m}[\mathbf{m}'/\mathbf{x}]) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} \} \\
\mathcal{V}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{ (\mathbf{W}, \boldsymbol{\Lambda} \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{val}}[\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]^{\boldsymbol{\Sigma}}_{\delta} \mid \\
 & & \quad \forall \mathbf{W}' \sqsupseteq \mathbf{W}, (\mathbf{t}', \mathbf{R}) \in \mathit{Rel}^{\hat{\uparrow}}{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}.(\mathbf{W}', \mathbf{m}[\mathbf{t}'/\boldsymbol{\alpha}]) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta[\boldsymbol{\alpha} \mapsto (\mathbf{t}', \mathbf{R})]} \} \\
\mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}\ \mathbf{t}']\!]^{\boldsymbol{\Sigma}}_{\delta} & = & (\mathcal{T}^{\hat{\uparrow}}[\![\mathbf{t} :: \boldsymbol{\kappa}' \to \boldsymbol{*}]\!]^{\boldsymbol{\Sigma}}_{\delta}\ (\delta_1(\mathbf{t}'), \mathcal{T}^{\hat{\uparrow}}[\![\mathbf{t}' :: \boldsymbol{\kappa}']\!]^{\boldsymbol{\Sigma}}_{\delta})) \\
\mathcal{E}^{\hat{\uparrow}}[\![\boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \delta_{\mathbf{R}}(\boldsymbol{\alpha}) \\
\mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{ (\mathbf{W}, \mathbf{m}) \in \mathit{Atom}^{\hat{\uparrow}}[\mathbf{t}]^{\boldsymbol{\Sigma}}_{\delta} \mid \\
 & & \quad \forall \mathbf{W}' \sqsupseteq \mathbf{W}, (\mathbf{W}', \mathbf{E}^{\not{}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}.(\mathbf{W}', \mathbf{E}^{\not{}}[\mathbf{m}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} \} \\
\mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{ (\mathbf{W}, \mathbf{E}^{\not{}}) \in \mathit{Atom}^{\hat{\uparrow}\mathrm{ctx}}[\mathbf{t}, |]^{\boldsymbol{\Sigma}}_{\delta} \\
 & & \quad \forall \mathbf{W}' \sqsupseteq \mathbf{W}, (\mathbf{W}', \mathbf{m}) \in \mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}.(\mathbf{W}', \mathbf{E}^{\not{}}[\mathbf{m}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta} \} \\
\mathcal{G}^{\hat{\uparrow}}[\![\cdot]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, \emptyset \,\} \\
\mathcal{G}^{\hat{\uparrow}}[\![\boldsymbol{\Gamma}, \mathbf{x} : \mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} & = & \{\, (\mathbf{W}, \boldsymbol{\gamma}[\mathbf{x} \mapsto \mathbf{m}]) \mid (\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\boldsymbol{\Gamma}]\!]^{\boldsymbol{\Sigma}}_{\delta} \wedge (\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta} \,\} \\
\mathcal{D}^{\hat{\uparrow}}[\![\cdot]\!]^{\boldsymbol{\Sigma}} & = & \{\, \emptyset \,\} \\
\mathcal{D}^{\hat{\uparrow}}[\![\boldsymbol{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa}]\!]^{\boldsymbol{\Sigma}} & = & \{\, \delta[\boldsymbol{\alpha} \mapsto (\mathbf{t}, \mathbf{R})] \mid \delta \in \mathcal{D}^{\hat{\uparrow}}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}} \wedge (\mathbf{t}, \mathbf{R}) \in \mathit{Rel}^{\hat{\uparrow}}{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \wedge \boldsymbol{\Sigma}_{\mathbf{D}} \vdash \mathbf{t} :: \boldsymbol{\kappa} \,\} \\
 & & \\
\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vDash^{\hat{\uparrow}} \mathbf{m} : \mathbf{t} & = & \forall \Gamma^{\hat{+}}_{\mathbf{1}}, \Gamma^{\hat{+}}_{\mathbf{2}}, \delta, \gamma, \boldsymbol{\gamma}. \\
 & & \Gamma^{\hat{+}}_{\mathbf{1}} \uplus \Gamma^{\hat{+}}_{\mathbf{2}} = \Gamma^{\hat{+}} \wedge \delta \in \mathcal{D}^{\hat{\uparrow}}[\![\boldsymbol{\Delta}]\!]^{\mathbf{D}_{\ell}} \wedge \\
 & & ((\mathbf{G_k}; \Gamma^{\hat{+}}_{\mathbf{2}}), \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}}_{\delta} \wedge \\
 & & ((\mathbf{G_k}; \Gamma^{\hat{+}}_{\mathbf{2}}), \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\boldsymbol{\Gamma}]\!]^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}}_{\delta} \implies \\
 & & ((\mathbf{G_k}; \Gamma^{\hat{+}}_{\mathbf{2}}), \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{D}_{\ell}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}}_{\delta}
\end{array}
$$

Figure 28: Back-translation: Logical Relations

5 Theorems

5.1 Noninterference

Theorem 5.1 (Noninterference)
If $\Gamma \vdash \mathsf{e} : \mathsf{s}$ *then* $\Gamma \vdash \mathsf{e} \approx_\zeta \mathsf{e} : \mathsf{s}$

Proof

By induction on $\Gamma \vdash \mathsf{e} : \mathsf{s}$. The case for bind is the only interesting case, which has two cases: if $\ell \sqsubseteq \zeta$, then the result follows by induction, otherwise the result follows from Lemma 5.2. □

Lemma 5.2
If $\ell \preceq \mathsf{s}$ *and* $\ell \not\sqsubseteq \zeta$ *then* $\forall(\mathsf{e}_1, \mathsf{e}_2) \in \mathit{Atom}\,[\mathsf{s}]\,.(\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta$

Proof

By induction on $\ell \preceq \mathsf{s}$. There are two interesting cases; the others follow easily by induction.

Show for arbitrary e_1, e_2, $(\mathsf{e}_1, \mathsf{e}_2) \in \mathit{Atom}\,[\mathsf{T}_{\ell'}\ \mathsf{s}]$ implies $(\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{T}_{\ell'}\ \mathsf{s}_1]\!]_\zeta$.

Since our language is terminating, we know $\mathsf{e}_1 \longmapsto^* \eta_{\ell'}\ \mathsf{e}_1'$ and $\mathsf{e}_2 \longmapsto^* \eta_{\ell'}\ \mathsf{e}_2'$, so it suffices to show $\ell' \sqsubseteq \zeta \implies (\mathsf{e}_1', \mathsf{e}_2')\mathcal{E}[\![\mathsf{s}_1]\!]_\zeta$.

Case P-Label1
We must show $(\mathsf{e}_1', \mathsf{e}_2') \in \mathcal{E}[\![\mathsf{s}_1]\!]_\zeta$. By assumption, $\ell \not\sqsubseteq \zeta$, and by P-Label1, we know $\ell \preceq \mathsf{s}_1$. By induction we know $\forall(\mathsf{e}_1, \mathsf{e}_2) \in \mathit{Atom}\,[\mathsf{s}_1]\,.(\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\mathsf{s}_1]\!]_\zeta$.

Case P-Label2
By assumption $\ell \not\sqsubseteq \zeta$, therefore $\ell' \not\sqsubseteq \zeta$. This immediately implies that $(\mathsf{e}_1, \mathsf{e}_2) \in \mathcal{V}[\![\mathsf{T}_\ell\ \mathsf{s}]\!]_\zeta$.

□

5.2 Transitivity and Symmetry

For technical reasons, we need to know the target relation is symmetric and transitive. Normally this is proved by showing the logical relation respects contextual equivalence. However, leaving the relation open with respect to $\mathbf{D}; \mathbf{G}$ makes proving this difficult or impossible.

Instead, we define functions sym() and trans(,) that compute symmetric and transitive versions of given relations, then extend these functions to $\boldsymbol{\rho}$s. We then prove the symmetric-transitive result of $[\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$ is itself $[\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$, yielding restricted, but strong enough, symmetry and transitivity lemmas.

5.2.1 Target Symmetry

$$
\begin{array}{rcl}
\mathrm{sym}(\mathbf{R} :: \boldsymbol{*}) & = & \{\, (\mathbf{m_1}, \mathbf{m_2}) \mid (\mathbf{m_2}, \mathbf{m_1}) \in \mathbf{R} \,\} \\
\mathrm{sym}(\mathbf{R} :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}) & = & \lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}').\mathrm{sym}(\mathbf{R}(\mathbf{t}_2, \mathbf{t}_1, \mathrm{sym}(\mathbf{R}'))) \\
\mathrm{sym}(\emptyset) & = & \emptyset \\
\mathrm{sym}(\boldsymbol{\rho}\,[\boldsymbol{\alpha} \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R})]) & = & \mathrm{sym}(\boldsymbol{\rho})\,[\boldsymbol{\alpha} \mapsto (\mathbf{t}_2, \mathbf{t}_1, \mathrm{sym}(\mathbf{R}))]
\end{array}
$$

Figure 29: F_ω: Symmetric Relations

Lemma 5.3 (sym Identity)
$\mathrm{sym}(\mathrm{sym}(\mathbf{R})) = \mathbf{R}$.

Lemma 5.4 (General Target Relation Symmetry)
If $\Delta \vdash \mathbf{t} :: \kappa$ *and* $\rho \in \mathcal{D}[\![\Delta]\!]^{\mathbf{D};\mathbf{G}}$ *then* $\mathrm{sym}(\mathcal{T}[\![\mathbf{t} :: \kappa]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{\kappa} \mathcal{T}[\![\mathbf{t} :: \kappa]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$

Proof

By induction on the derivation $\Delta \vdash \mathbf{t} :: \kappa$.

Case $\text{(FK-Unit)}\ \dfrac{}{\Delta \vdash \mathbf{1} :: *}$

We must show $\mathrm{sym}(\mathcal{T}[\![\mathbf{1} :: *]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{*} \mathcal{T}[\![\mathbf{1} :: *]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

It suffices to show $\mathrm{sym}(\mathcal{V}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{*} \mathcal{V}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

By definition, it suffices to show, for all $(\mathbf{m_2}, \mathbf{m_1}) \in \mathcal{V}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$, that $(\mathbf{m_1}, \mathbf{m_2}) \in \mathcal{V}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$, which follows by definition of $\mathcal{V}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$.

Case $\text{(FK-Var)}\ \dfrac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa}$

We must show $\mathrm{sym}(\mathcal{T}[\![\alpha :: \kappa]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{\kappa} \mathcal{T}[\![\alpha :: *]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

By definition it suffices to show $\mathrm{sym}(\rho(\alpha)) = \mathrm{sym}(\rho)(\alpha)$, which follows by definition of sym().

Case $\text{(FK-Pair)}\ \dfrac{\Delta \vdash \mathrm{t}_1 :: * \qquad \Delta \vdash \mathrm{t}_2 :: *}{\Delta \vdash \mathrm{t}_1 \times \mathrm{t}_2 :: *}$

Follows from the IH.

Case $\text{(FK-Sum)}\ \dfrac{\Delta \vdash \mathrm{t}_1 :: * \qquad \Delta \vdash \mathrm{t}_2 :: *}{\Delta \vdash \mathrm{t}_1 + \mathrm{t}_2 :: *}$

Follows from the IH.

Case $\text{(FK-Arrow)}\ \dfrac{\Delta \vdash \mathrm{t}_1 :: * \qquad \Delta \vdash \mathrm{t}_2 :: *}{\Delta \vdash \mathrm{t}_1 \rightarrow \mathrm{t}_2 :: *}$

It suffices to show $\mathrm{sym}(\mathcal{V}[\![\mathrm{t}_1 \rightarrow \mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{*} \mathcal{V}[\![\mathrm{t}_1 \rightarrow \mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

We show only one direction, since the other is exactly symmetric.

For arbitrary $(\lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_2}, \lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_1}) \in \mathcal{V}[\![\mathrm{t}_1 \rightarrow \mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$, we must show $(\lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_1}, \lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_2}) \in \mathcal{V}[\![\mathrm{t}_1 \rightarrow \mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

For arbitrary $(\mathbf{m}, \mathbf{m}') \in \mathcal{E}[\![\mathrm{t}_1]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$, that $(\mathbf{m_1}[\mathrm{x}/\mathbf{m}], \mathbf{m_2}[\mathrm{x}/\mathbf{m}']) \in \mathcal{E}[\![\mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

By the IH applied to $\Delta \vdash \mathrm{t}_2 :: *$, it suffices to show $(\mathbf{m_2}[\mathrm{x}/\mathbf{m}'], \mathbf{m_1}[\mathrm{x}/\mathbf{m}]) \in \mathcal{E}[\![\mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$.

By the IH applied to $\Delta \vdash \mathrm{t}_1 :: *$, we know $\mathrm{sym}(\mathcal{E}[\![\mathrm{t}_1]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{*} \mathcal{E}[\![\mathrm{t}_1]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$ and therefore $(\mathbf{m}', \mathbf{m}) \in \mathcal{E}[\![\mathrm{t}_1]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$.

Instantiating $(\lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_2}, \lambda \mathrm{x}{:}\mathrm{t}_1.\mathbf{m_1}) \in \mathcal{V}[\![\mathrm{t}_1 \rightarrow \mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$ with $(\mathbf{m}', \mathbf{m})$, we conclude $(\mathbf{m_2}[\mathrm{x}/\mathbf{m}'], \mathbf{m_1}[\mathrm{x}/\mathbf{m}]) \in \mathcal{E}[\![\mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$.

Case $\text{(FK-Abs)}\ \dfrac{\Delta, \alpha :: \kappa \vdash \mathbf{t} :: *}{\Delta \vdash \forall \alpha{::}\kappa.\,\mathbf{t} :: *}$

It suffices to show $\mathrm{sym}(\mathcal{V}[\![\forall \alpha{::}\kappa.\,\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}) \equiv^{\mathbf{D};\mathbf{G}}_{*} \mathcal{V}[\![\forall \alpha{::}\kappa.\,\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

We show only one direction, since the other is exactly symmetric.

For arbitrary $(\Lambda \alpha{::}\kappa.\mathbf{m_2}, \Lambda \alpha{::}\kappa.\mathbf{m_1}) \in \mathcal{V}[\![\forall \alpha{::}\kappa.\,\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$, we must show $(\Lambda \alpha{::}\kappa.\mathbf{m_1}, \Lambda \alpha{::}\kappa.\mathbf{m_2}) \in \mathcal{V}[\![\forall \alpha{::}\kappa.\,\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)}$.

For arbitrary $\pi \in Rel^{\mathbf{D};\mathbf{G}}_{\kappa}$, we must show $(\mathbf{m_1}[\alpha/\pi_1], \mathbf{m_2}[\alpha/\pi_2]) \in \mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{sym}(\rho)[\alpha \mapsto \pi]}$.

Let $\pi' = (\pi_2, \pi_1, \mathrm{sym}(\pi_{\mathbf{R}}))$.

Instantiating $(\Lambda\alpha{::}\kappa.\mathbf{m_2}, \Lambda\alpha{::}\kappa.\mathbf{m_1}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\,\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$ with π' we know $(\mathbf{m_2}[\alpha/\pi_2], \mathbf{m_1}[\alpha/\pi_1]) \in \mathcal{E}[\![\mathbf{t}]\!]_{\rho[\alpha\mapsto\pi']}^{\mathbf{D};\mathbf{G}}$.

Therefore, it suffices to show $\mathrm{sym}(\mathcal{E}[\![\mathbf{t}]\!]_{\rho[\alpha\mapsto\pi']}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{E}[\![\mathbf{t}]\!]_{\mathrm{sym}(\rho)[\alpha\mapsto\pi]}^{\mathbf{D};\mathbf{G}}$, which follows from the IH applied to $\mathbf{\Delta}, \alpha :: \kappa \vdash \mathbf{t} :: *$, with $\rho\,[\alpha \mapsto (\pi_2, \pi_1, \mathrm{sym}(\pi_{\mathbf{R}}))]$.

Case $$\text{(FK-Fun)}\ \frac{\mathbf{\Delta}, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2}{\mathbf{\Delta} \vdash \lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2}$$

We must show $\mathrm{sym}(\mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \to \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$.

By definition, it suffices to show
$\lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}').\mathrm{sym}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\rho[\alpha\mapsto(\mathbf{t}_2,\mathbf{t}_1,\mathrm{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \to \kappa_2}^{\mathbf{D};\mathbf{G}} \lambda_R(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}').\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\mathrm{sym}(\rho)[\alpha\mapsto(\mathbf{t}_1,\mathbf{t}_2,\mathbf{R}')]}^{\mathbf{D};\mathbf{G}}$.

For arbitrary $\pi \in Rel_{\kappa}^{\mathbf{D};\mathbf{G}}$, it suffices to show
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\rho[\alpha\mapsto(\pi_2,\pi_1,\mathrm{sym}(\pi_{\mathbf{R}}))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\mathrm{sym}(\rho[\alpha\mapsto(\mathbf{t}_2,\mathbf{t}_1,\mathrm{sym}(\mathbf{R}'))])}^{\mathbf{D};\mathbf{G}}$.

By the IH applied to $\mathbf{\Delta}, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$, we know
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\rho[\alpha\mapsto(\mathbf{t}_2,\mathbf{t}_1,\mathrm{sym}(\mathbf{R}'))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\mathrm{sym}(\rho[\alpha\mapsto(\mathbf{t}_2,\mathbf{t}_1,\mathrm{sym}(\mathbf{R}'))])}^{\mathbf{D};\mathbf{G}}$

By definition and Lemma 5.3, we know $\mathrm{sym}(\rho\,[\alpha \mapsto (\mathbf{t}_2, \mathbf{t}_1, \mathrm{sym}(\mathbf{R}'))]) = \mathrm{sym}(\rho)\,[\alpha \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}')]$, therefore
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\rho[\alpha\mapsto(\pi_2,\pi_1,\mathrm{sym}(\pi_{\mathbf{R}}))]}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]_{\mathrm{sym}(\rho[\alpha\mapsto(\mathbf{t}_2,\mathbf{t}_1,\mathrm{sym}(\mathbf{R}'))])}^{\mathbf{D};\mathbf{G}}$.

Case $$\text{(FK-App)}\ \frac{\mathbf{\Delta} \vdash \mathbf{t}_1 :: \kappa_1 \to \kappa_2 \qquad \mathbf{\Delta} \vdash \mathbf{t}_2 :: \kappa_1}{\mathbf{\Delta} \vdash \mathbf{t}_1\ \mathbf{t}_2 :: \kappa_2}$$

We must show $\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1\ \mathbf{t}_2 :: \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1\ \mathbf{t}_2 :: \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$.

Let $\pi = (\rho_1\ \mathbf{t}_2, \rho_2\ \mathbf{t}_2, \mathcal{T}[\![\mathbf{t}_2 :: \kappa_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}})$ and $\pi' = (\rho_2\ \mathbf{t}_2, \rho_1\ \mathbf{t}_2, \mathrm{sym}(\mathcal{T}[\![\mathbf{t}_2 :: \kappa_1]\!]_{\rho}^{\mathbf{D};\mathbf{G}}))$.

By definition of $\mathcal{T}[\![\mathbf{t}_1\ \mathbf{t}_2 :: \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$, it suffices to show
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\ \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}\ (\pi_1, \pi_2, \mathcal{T}[\![\mathbf{t}_2 :: \kappa_1]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}})$.

By the IH applied to $\mathbf{\Delta} \vdash \mathbf{t}_2 :: \kappa_1$, it suffices to show $\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\ \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}\ \pi'$.

By the IH applied to $\mathbf{\Delta} \vdash \mathbf{t}_1 :: \kappa_1 \to \kappa_2$ we know
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{\kappa_1 \to \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$.

By definition of $\equiv_{\kappa_1 \to \kappa_2}^{\mathbf{D};\mathbf{G}}$, we know
$\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}})\ \pi' \equiv_{\kappa_1 \to \kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}\ \pi'$.

By definition, we know $\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}) = \lambda_R(\mathbf{t}_1', \mathbf{t}_2', \mathbf{R}').\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\ (\mathbf{t}_2', \mathbf{t}_1', \mathrm{sym}(\mathbf{R}')))$

Therefore, $\mathrm{sym}(\mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\rho}^{\mathbf{D};\mathbf{G}}\ \pi) \equiv_{\kappa_2}^{\mathbf{D};\mathbf{G}} \mathcal{T}[\![\mathbf{t}_1 :: \kappa_1 \to \kappa_2]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}\ \pi'$.

$\square$

Lemma 5.5 (Lattice Interp. Self-Symmetric)
Let $\rho = [\![\mathcal{L}_{\ell}^{+}]\!]_{\zeta}^{\mathbf{\Sigma}}$. $\mathrm{sym}(\rho) = \rho$

Lemma 5.6 (Target Expression Relation Symmetry)
If $(\mathbf{m_1}, \mathbf{m_2}) \in \mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}$ *then* $(\mathbf{m_2}, \mathbf{m_1}) \in \mathcal{E}[\![\mathbf{t}]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$

Proof

We must show $(\mathbf{m_2}, \mathbf{m_1}) \in \mathcal{E}[\![\mathbf{t}]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$

By Lemma 5.7 (General Target Relation Symmetry), we know $\mathrm{sym}(\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}}) \equiv_{*}^{\mathbf{D};\mathbf{G}} \mathcal{E}[\![\mathbf{t}]\!]_{\mathrm{sym}(\rho)}^{\mathbf{D};\mathbf{G}}$.

By definition of $\equiv_{*}^{\mathbf{D};\mathbf{G}}$, it suffices to show $(\mathbf{m_2}, \mathbf{m_1}) \in \mathrm{sym}(\mathcal{E}[\![\mathbf{t}]\!]_{\rho}^{\mathbf{D};\mathbf{G}})$

By definition of $\mathrm{sym}(\mathbf{R} :: *)$, $(\mathbf{m_2}, \mathbf{m_1}) \in \mathrm{sym}(\mathbf{R})$ if $(\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R}$.

Take $\mathbf{R}$ to be $\boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$

It suffices to show $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$, which follows by assumption. □

Lemma 5.7 (Target Relation Symmetric under Lattice Interp.)
Let $\boldsymbol{\rho} = [\![\mathcal{L}^{+}_{\ell}]\!]^{\boldsymbol{\Sigma}}_{\zeta}$.
If $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ *then* $(\mathbf{m_2}, \mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

5.2.2 Target Transitivity

$$
\begin{array}{rcl}
\mathrm{fst}_{\mathbf{R}}(\mathbf{R})_{*} & = & \{(\mathbf{m_1}, \langle\rangle) \mid (\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R}\} \\
\mathrm{fst}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}} & = & \lambda_R \boldsymbol{\pi}.\mathrm{fst}_{\mathbf{R}}(\mathbf{R}\ \boldsymbol{\pi})_{\boldsymbol{\kappa_2}} \\
\mathrm{snd}_{\mathbf{R}}(\mathbf{R})_{*} & = & \{(\langle\rangle, \mathbf{m_2}) \mid (\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R}\} \\
\mathrm{snd}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}} & = & \lambda_R \boldsymbol{\pi}.\mathrm{snd}_{\mathbf{R}}(\mathbf{R}\ \boldsymbol{\pi})_{\boldsymbol{\kappa_2}} \\
\mathbf{1}_{*} & = & \mathbf{1} \\
\mathbf{1}_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}} & = & \boldsymbol{\lambda\alpha{::}\kappa_1.1_{\kappa_2}}
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{trans}(\mathbf{R_1}, \mathbf{R_2})_{*} & = & \{(\mathbf{m_1}, \mathbf{m_3}) \mid (\mathbf{m_1}, \mathbf{m_2}) \in \mathbf{R_1} \wedge (\mathbf{m_2}, \mathbf{m_3}) \in \mathbf{R_2}\} \\
\mathrm{trans}(\mathbf{R_1}, \mathbf{R_2})_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}} & = & \lambda_R(\mathbf{t}_1, \mathbf{t}_3, \mathbf{R}).\mathrm{trans}((\mathbf{R_1}\ (\mathbf{t}_1, \mathbf{1}_{\boldsymbol{\kappa_1}}, \mathrm{fst}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa_1}})), (\mathbf{R_2}\ (\mathbf{1}_{\boldsymbol{\kappa_1}}, \mathbf{t}_2, \mathrm{snd}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa_1}})))_{\boldsymbol{\kappa_2}}
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{trans}(\emptyset, \emptyset) & = & \emptyset \\
\mathrm{trans}(\boldsymbol{\rho}\,[\boldsymbol{\alpha} :: \boldsymbol{\kappa} \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_1)]\,, \boldsymbol{\rho}'\,[\boldsymbol{\alpha} :: \boldsymbol{\kappa} \mapsto (\mathbf{t}_2, \mathbf{t}_3, \mathbf{R_2})]) & = & (\mathrm{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}'))\,[\boldsymbol{\alpha} \mapsto (\mathbf{t}_1, \mathbf{t}_3, \mathrm{trans}(\mathbf{R_1}, \mathbf{R_2})_{\boldsymbol{\kappa}})]
\end{array}
$$

Figure 30: F_{ω}: Transitive Relations

Lemma 5.8 (trans Identity)
$\mathrm{trans}(\mathrm{fst}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa}}, \mathrm{snd}_{\mathbf{R}}(\mathbf{R})_{\boldsymbol{\kappa}})_{\boldsymbol{\kappa}} = \mathbf{R}$

Lemma 5.9 (fst Identity)
$\mathrm{fst}_{\mathbf{R}}(\mathrm{trans}(\mathbf{R_1}, \mathbf{R_2})_{\boldsymbol{\kappa}})_{\boldsymbol{\kappa}} = \mathbf{R_1}$

Lemma 5.10 (snd Identity)
$\mathrm{snd}_{\mathbf{R}}(\mathrm{trans}(\mathbf{R_1}, \mathbf{R_2})_{\boldsymbol{\kappa}})_{\boldsymbol{\kappa}} = \mathbf{R_2}$

Lemma 5.11 (General Target Relation Transitivity)
If $\boldsymbol{\Delta} \vdash \mathbf{t} :: \boldsymbol{\kappa}$, $\boldsymbol{\rho} \in \boldsymbol{\mathcal{D}}[\![\boldsymbol{\Delta}]\!]^{\mathbf{D};\mathbf{G}}$, $\boldsymbol{\rho}' \in \boldsymbol{\mathcal{D}}[\![\boldsymbol{\Delta}]\!]^{\mathbf{D};\mathbf{G}}$, *and* $\boldsymbol{\rho_2} = \boldsymbol{\rho}'_1$, *then*
$\mathrm{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{T}}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}'})_{\boldsymbol{\kappa}} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa}} \boldsymbol{\mathcal{T}}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}')}$.

Proof

By induction on the derivation $\boldsymbol{\Delta} \vdash \mathbf{t} :: \boldsymbol{\kappa}$.

Case $\text{(FK-Unit)}\ \dfrac{}{\boldsymbol{\Delta} \vdash \mathbf{1} :: *}$

It suffices to show $\mathrm{trans}(\boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}'})_{*} \equiv^{\mathbf{D};\mathbf{G}}_{*} \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}')}$.

For arbitrary $(\mathbf{m_1}, \mathbf{m_3}) \in \mathrm{trans}(\boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}'})_{*}$, we must show $(\mathbf{m_1}, \mathbf{m_3}) \in \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}')}$.

By definition, $(\mathbf{m_1}, \mathbf{m_3}) \in \mathrm{trans}(\boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}'})_{*}$ implies $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$ and $(\mathbf{m_2}, \mathbf{m_3}) \in \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}'}$.

Therefore $(\mathbf{m_1}, \mathbf{m_3}) = (\langle\rangle, \langle\rangle)$, thus $(\mathbf{m_1}, \mathbf{m_3}) \in \boldsymbol{\mathcal{V}}[\![\mathbf{1}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}')}$.

Case (FK-Var) $\dfrac{\alpha :: \kappa \in \Delta}{\Delta \vdash \alpha :: \kappa}$

It suffices to show $\mathrm{trans}(\rho_{\mathrm{R}}(\alpha), \rho'_{\mathrm{R}}(\alpha))_\kappa \equiv_\kappa^{D;G} \mathrm{trans}(\rho, \rho')_{\kappa\,\mathrm{R}}(\alpha)$, which follows by definition of $\mathrm{trans}(\rho, \rho')_\kappa$.

Case (FK-Pair) $\dfrac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \times t_2 :: *}$

It suffices to show $\mathrm{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{D;G})_* \equiv_*^{D;G} \mathcal{V}[\![t_1 \times t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

For arbitrary $(m_1, m_3) \in \mathrm{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{D;G})_*$, we must show $(m_1, m_3) \in \mathcal{V}[\![t_1 \times t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

By definition of $\mathrm{trans}(\mathcal{V}[\![t_1 \times t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{D;G})_*$, we know $(m_1, m_2) \in \mathcal{V}[\![t_1 \times t_2]\!]_\rho^{D;G}$ and $(m_2, m_3) \in \mathcal{V}[\![t_1 \times t_2]\!]_{\rho'}^{D;G}$.
Therefore, $m_1 = \langle m'_1, m''_1\rangle$, $m_2 = \langle m'_2, m''_2\rangle$, and $m_3 = \langle m'_3, m''_3\rangle$.
So it suffices to show $(m'_1, m'_3) \in \mathcal{E}[\![t_1]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$ and $(m''_1, m''_3) \in \mathcal{E}[\![t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

We know $(m'_1, m'_2) \in \mathcal{E}[\![t_1]\!]_\rho^{D;G}$ and $(m'_2, m'_3) \in \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G}$. By definition, therefore, $(m'_1, m'_3) \in \mathrm{trans}(\mathcal{E}[\![t_1]\!]_\rho^{D;G}, \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G})_*$.

Similarly we know $(m''_1, m''_3) \in \mathrm{trans}(\mathcal{E}[\![t_2]\!]_\rho^{D;G}, \mathcal{E}[\![t_2]\!]_{\rho'}^{D;G})_*$.

By IH applied to $\Delta \vdash t_1 :: *$, we know $(m'_1, m'_3) \in \mathcal{E}[\![t_1]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

By IH applied to $\Delta \vdash t_2 :: *$, we know $(m''_1, m''_3) \in \mathcal{E}[\![t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

Case (FK-Sum) $\dfrac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 + t_2 :: *}$

Follows by IH, similar to the previous case.

Case (FK-Arrow) $\dfrac{\Delta \vdash t_1 :: * \qquad \Delta \vdash t_2 :: *}{\Delta \vdash t_1 \to t_2 :: *}$

It suffices to show $\mathrm{trans}(\mathcal{V}[\![t_1 \to t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \to t_2]\!]_{\rho'}^{D;G})_* \equiv_*^{D;G} \mathcal{V}[\![t_1 \to t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

For arbitrary $(m_1, m_3) \in \mathrm{trans}(\mathcal{V}[\![t_1 \to t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \to t_2]\!]_{\rho'}^{D;G})_*$, we must show $(m_1, m_3) \in \mathcal{V}[\![t_1 \to t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

By definition of $\mathrm{trans}(\mathcal{V}[\![t_1 \to t_2]\!]_\rho^{D;G}, \mathcal{V}[\![t_1 \to t_2]\!]_{\rho'}^{D;G})_*$, we know $(m_1, m_2) \in \mathcal{V}[\![t_1 \to t_2]\!]_\rho^{D;G}$ and $(m_2, m_3) \in \mathcal{V}[\![t_1 \to t_2]\!]_{\rho'}^{D;G}$.
Therefore, $m_1 = \lambda x{:}t_1.m'_1$, $m_2 = \lambda x{:}t_1.m'_2$, and $m_3 = \lambda x{:}t_1.m'_3$.
For arbitrary $(m, m'') \in \mathcal{E}[\![t_1]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$, it suffices to show $(m'_1[m/x], m'_3[m''/x]) \in \mathcal{E}[\![t_2]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

By IH applied to $\Delta \vdash t_2 :: *$, it suffices to show $(m'_1[m/x], m'_3[m''/x]) \in \mathrm{trans}(\mathcal{E}[\![t_2]\!]_\rho^{D;G}, \mathcal{E}[\![t_2]\!]_{\rho'}^{D;G})_*$.

By IH applied to $\Delta \vdash t_1 :: *$, we know $\mathrm{trans}(\mathcal{E}[\![t_1]\!]_\rho^{D;G}, \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G}) \equiv_*^{D;G} \mathcal{E}[\![t_1]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

Therefore, $(m, m'') \in \mathrm{trans}(\mathcal{E}[\![t_1]\!]_\rho^{D;G}, \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G})$.

By definition of $\mathrm{trans}(\mathcal{E}[\![t_1]\!]_\rho^{D;G}, \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G})$, $(m, m') \in \mathcal{E}[\![t_1]\!]_\rho^{D;G}$ and $(m', m'') \in \mathcal{E}[\![t_1]\!]_{\rho'}^{D;G}$.

Therefore, $(m'_1[m/x], m'_2[m'/x]) \in \mathcal{V}[\![t_2]\!]_\rho^{D;G}$ and $(m'_2[m'/x], m'_3[m''/x]) \in \mathcal{V}[\![t_2]\!]_{\rho'}^{D;G}$.

Therefore, by definition, $(m'_1[m/x], m'_3[m''/x]) \in \mathrm{trans}(\mathcal{E}[\![t_2]\!]_\rho^{D;G}, \mathcal{E}[\![t_2]\!]_{\rho'}^{D;G})_*$.

Case (FK-Abs) $\dfrac{\Delta, \alpha :: \kappa \vdash t :: *}{\Delta \vdash \forall \alpha{::}\kappa.\, t :: *}$

It suffices to show $\mathrm{trans}(\mathcal{V}[\![\forall \alpha{::}\kappa.\, t]\!]_\rho^{D;G}, \mathcal{V}[\![\forall \alpha{::}\kappa.\, t]\!]_{\rho'}^{D;G})_* \equiv_*^{D;G} \mathcal{V}[\![\forall \alpha{::}\kappa.\, t]\!]_{\mathrm{trans}(\rho,\rho')}^{D;G}$.

For arbitrary $(\mathbf{m_1}, \mathbf{m_3}) \in \mathrm{trans}(\mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_*$, we must show $(\mathbf{m_1}, \mathbf{m_3}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$.

By definition of $\mathrm{trans}(\mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_*$, we know $(\mathbf{m_1}, \mathbf{m_2}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$ and $(\mathbf{m_2}, \mathbf{m_3}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho'}$.

Therefore, $\mathbf{m_1} = \mathbf{\Lambda}\alpha{::}\kappa.\mathbf{m}'_1$, $\mathbf{m_2} = \mathbf{\Lambda}\alpha{::}\kappa.\mathbf{m}'_2$, and $\mathbf{m_3} = \mathbf{\Lambda}\alpha{::}\kappa.\mathbf{m}'_3$.

For arbitrary $\pi \in Rel^{\mathbf{D};\mathbf{G}}_{\kappa}$, we must show $(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')[\alpha::\kappa\mapsto\pi]}$.

Let $\pi' = (\pi_1, \mathbf{1}_\kappa, \mathrm{fst}_{\mathbf{R}}(\pi_{\mathbf{R}})_\kappa)$ and $\pi'' = (\mathbf{1}_\kappa, \pi_2, \mathrm{snd}_{\mathbf{R}}(\pi_{\mathbf{R}})_\kappa)$.

Note that, by Lemma 5.8, $\mathrm{trans}(\rho\,[\alpha \mapsto \pi'], \rho\,[\alpha \mapsto \pi'']) = \mathrm{trans}(\rho, \rho')\,[\alpha \mapsto \pi]$.

By IH applied to $\mathbf{\Delta}, \alpha :: \kappa \vdash \mathbf{t} :: *$, we know $\mathrm{trans}(\mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho[\alpha\mapsto\pi']}, \mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho'[\alpha\mapsto\pi'']})_* \equiv^{\mathbf{D};\mathbf{G}}_* \mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')[\alpha\mapsto\pi]}$.

Therefore, to show $(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')[\alpha::\kappa\mapsto\pi]}$, it suffices to show

$(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathrm{trans}(\mathcal{E}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho[\alpha\mapsto\pi']}, \mathcal{E}[\![\rho'\,[\alpha \mapsto \pi'']]\!]^{\mathbf{D};\mathbf{G}}_{\rho})_*$.

By definition, it suffices to show

$(\mathbf{m}'_1[\pi_1/\alpha], \mathbf{m}'_2[\mathbf{1}_\kappa/\alpha]) \in \mathcal{E}[\![\mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho[\alpha\mapsto\pi']}$ and

$(\mathbf{m}'_2[\mathbf{1}_\kappa/\alpha], \mathbf{m}'_3[\pi_2/\alpha]) \in \mathcal{E}[\![\mathrm{t}_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'[\alpha\mapsto\pi'']}$,

which follow from instantiating $(\mathbf{m_1}, \mathbf{m_2}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho}$ with π' and $(\mathbf{m_2}, \mathbf{m_3}) \in \mathcal{V}[\![\forall\alpha{::}\kappa.\, \mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\rho'}$ with π''.

Case (FK-Fun) $\dfrac{\mathbf{\Delta}, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2}{\mathbf{\Delta} \vdash \lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2}$

We must show

$\mathrm{trans}(\mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_{\kappa_1 \to \kappa_2} \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_1 \to \kappa_2}$
$\mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$.

By definition, it suffices to show

$\lambda_R(\mathrm{t}_1, \mathrm{t}_3, \mathbf{R}).\mathrm{trans}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}\rho\,[\alpha :: \kappa_1 \mapsto (\mathrm{t}_1, \mathbf{1}_{\kappa_1}, \mathrm{fst}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})],$
$\quad \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}\rho\,[\alpha :: \kappa_1 \mapsto (\mathbf{1}_{\kappa_1}, \mathrm{t}_2, \mathrm{snd}_{\mathbf{R}}(\mathbf{R})_{\kappa_1})])_{\kappa_2}$
$\equiv^{\mathbf{D};\mathbf{G}}_{\kappa_1 \to \kappa_2} \mathcal{T}[\![\lambda\alpha{::}\kappa.\mathbf{t} :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$.

For arbitrary $\pi \in Rel^{\mathbf{D};\mathbf{G}}_{\kappa}\kappa_1$, it suffices to show

$\mathrm{trans}(\mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho[\alpha::\kappa_1\mapsto(\pi_1,\mathbf{1}_{\kappa_1},\mathrm{fst}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa_1})]}, \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'[\alpha::\kappa_1\mapsto(\mathbf{1}_{\kappa_1},\pi_2,\mathrm{snd}_{\mathbf{R}}(\pi_{\mathbf{R}})_{\kappa_1})]})_{\kappa_2} \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_1 \to \kappa_2} \mathcal{T}[\![\mathbf{t} :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$

which follows by IH applied to $\mathbf{\Delta}, \alpha :: \kappa_1 \vdash \mathbf{t} :: \kappa_2$, and Lemma 5.8.

Case (FK-App) $\dfrac{\mathbf{\Delta} \vdash \mathrm{t}_1 :: \kappa_1 \to \kappa_2 \qquad \mathbf{\Delta} \vdash \mathrm{t}_2 :: \kappa_1}{\mathbf{\Delta} \vdash \mathrm{t}_1\ \mathrm{t}_2 :: \kappa_2}$

We must show $\mathrm{trans}(\mathcal{T}[\![\mathrm{t}_1\ \mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{T}[\![\mathrm{t}_1\ \mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_{\kappa_2} \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_2} \mathcal{T}[\![\mathrm{t}_1\ \mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$.

Let $\pi = (\mathrm{trans}(\rho, \rho')\ _1\ \mathrm{t}_2, \mathrm{trans}(\rho, \rho')\ _2\ \mathrm{t}_2, \mathcal{T}[\![\mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')})$, $\pi' = (\rho_1\ \mathrm{t}_2, \rho_2\ \mathrm{t}_2, \mathcal{T}[\![\mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho})$, and $\pi'' = (\rho'_1\ \mathrm{t}_2, \rho'_2\ \mathrm{t}_2, \mathcal{T}[\![\mathrm{t}_2 :: \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})$.

By definition, it suffices to show $\mathrm{trans}(\mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}\ \pi', \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'}\ \pi'')_{\kappa_2} \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_2}$
$\mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}\ \pi$

By the IH applied to $\mathbf{\Delta} \vdash \mathrm{t}_1 :: \kappa_1 \to \kappa_2$ we know

$\mathrm{trans}(\mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_{\kappa_1 \to \kappa_2} \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_1 \to \kappa_2} \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}$

By definition of $\equiv^{\mathbf{D};\mathbf{G}}_{\kappa_1 \to \kappa_2}$, we know

$\mathrm{trans}(\mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_{\kappa_1 \to \kappa_2}\ \pi \equiv^{\mathbf{D};\mathbf{G}}_{\kappa_2} \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\mathrm{trans}(\rho,\rho')}\ \pi$

By definition, we know

$\mathrm{trans}(\mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho}, \mathcal{T}[\![\mathrm{t}_1 :: \kappa_1 \to \kappa_2]\!]^{\mathbf{D};\mathbf{G}}_{\rho'})_{\kappa_1 \to \kappa_2} =$

$\lambda_R(\mathbf{t}, \mathbf{t'}, \mathbf{R}).\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ (\mathbf{t}, \mathbf{1}_{\boldsymbol{\kappa_2}}, \text{fst}_{\mathbf{R}}(\mathbf{R})), \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ (\mathbf{1}_{\boldsymbol{\kappa_2}}, \mathbf{t'}, \text{snd}_{\mathbf{R}}(\mathbf{R})))_{\boldsymbol{\kappa_2}}$
Therefore,
$\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ (\boldsymbol{\pi_1}, \mathbf{1}_{\boldsymbol{\kappa_2}}, \text{fst}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}})), \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ (\mathbf{1}_{\boldsymbol{\kappa_2}}, \boldsymbol{\pi_2}, \text{snd}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}})))_{\boldsymbol{\kappa_2}} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa_2}}$
$\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\text{trans}(\boldsymbol{\rho},\boldsymbol{\rho'})}\ \boldsymbol{\pi}$
It suffices to show
$\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ (\boldsymbol{\pi_1}, \mathbf{1}_{\boldsymbol{\kappa_2}}, \text{fst}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}})), \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ (\mathbf{1}_{\boldsymbol{\kappa_2}}, \boldsymbol{\pi_2}, \text{snd}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}})))_{\boldsymbol{\kappa_2}} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa_2}}$
$\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ \boldsymbol{\pi'}, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ \boldsymbol{\pi''})_{\boldsymbol{\kappa_2}}$.
By IH applied to $\boldsymbol{\Delta} \vdash \mathbf{t}_2 :: \boldsymbol{\kappa_2}$, we know $\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}})_{\boldsymbol{\kappa_2}} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa_2}} \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\text{trans}(\boldsymbol{\rho},\boldsymbol{\rho'})}$.
Therefore, by Lemma 5.9, we know $\text{fst}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}}) = \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$, and by Lemma 5.10 we know $\text{snd}_{\mathbf{R}}(\boldsymbol{\pi}_{\mathbf{R}}) = \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}$.
Since trans ignores the positions $\mathbf{1}_{\boldsymbol{\kappa_2}}$, $\boldsymbol{\pi'_2}$, and $\boldsymbol{\pi''_1}$, it follows that

$$\begin{aligned}
&\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ (\boldsymbol{\pi_1}, \mathbf{1}_{\boldsymbol{\kappa_2}}, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}),\\
&\qquad \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ (\mathbf{1}_{\boldsymbol{\kappa_2}}, \boldsymbol{\pi_2}, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}))_{\boldsymbol{\kappa_2}} \qquad \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa_2}}\\
&\text{trans}(\boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}\ \boldsymbol{\pi'}, \boldsymbol{\mathcal{T}}[\![\mathbf{t}_1 :: \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho'}}\ \boldsymbol{\pi''})_{\boldsymbol{\kappa_2}}.
\end{aligned}$$

□

Lemma 5.12 (Lattice Interp. Self-Trans)
Let $\boldsymbol{\rho} = [\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_\zeta$. $\text{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}) = \boldsymbol{\rho}$

Lemma 5.13 (Target Relation Transitivity under Lattice Interp.)
Let $\boldsymbol{\rho} = [\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_\zeta$.
If $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ *and* $(\mathbf{m_2}, \mathbf{m_3}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ *then* $(\mathbf{m_1}, \mathbf{m_3}) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

Proof

By Lemma 5.11, we know $\text{trans}(\boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}})_{\boldsymbol{*}} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{*}} \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\text{trans}(\boldsymbol{\rho},\boldsymbol{\rho})}$.

By Lemma 5.12, we know $\text{trans}(\boldsymbol{\rho}, \boldsymbol{\rho}) = \boldsymbol{\rho}$, therefore $\text{trans}(\boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}})_{\boldsymbol{*}} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{*}} \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

The result follows by definition of $\text{trans}(\boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}, \boldsymbol{\mathcal{E}}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}})_{\boldsymbol{*}}$. □

5.3 Parametricity

Theorem 5.14 (Fundamental Property)
If $\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}$ *then* $\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} \approx \mathbf{m} : \mathbf{t}$

Proof

The structure of proof is similar to (but less complex than) Lemma 5.34 (Fundamental Property of Back-translation), and the work of Vytiniotis and Weirich [4].

We omit the statement of key lemmas as they are already well explained by Vytiniotis and Weirich [4].

□

Lemma 5.15 (Preservation)
If $\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}$ *and* $\mathbf{m} \longmapsto^* \mathbf{m'}$ *then* $\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m'} : \mathbf{t}$

Lemma 5.16 (Canonical Forms)
Let $\boldsymbol{\Delta}; \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}$.

- *If* $\mathbf{t} = \boldsymbol{\alpha} \in \boldsymbol{\Delta}$, $\mathbf{m} \longmapsto^* \mathbf{m'}$ *and* $\text{irred}(\mathbf{m'})$.
- *If* $\mathbf{t} = \mathbf{1}$, $\mathbf{m} \longmapsto^* \boldsymbol{\langle\rangle}$.

- *If* $\mathbf{t} = \mathbf{t}_1 \times \mathbf{t}_2$, $\mathbf{m} \longmapsto^* \langle \mathbf{m_1}, \mathbf{m_2} \rangle$.
- *If* $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$, $\mathbf{m} \longmapsto^* \mathbf{inj_i}\ \mathbf{m}'$.
- *If* $\mathbf{t} = \mathbf{t}_1 \to \mathbf{t}_2$, $\mathbf{m} \longmapsto^* \boldsymbol{\lambda}\mathbf{x}{:}\mathbf{t}_1.\mathbf{m}'$.
- *If* $\mathbf{t} = \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}'$, $\mathbf{m} \longmapsto^* \boldsymbol{\Lambda\alpha}{::}{*}.\mathbf{m}'$.

Proof

Follows by Theorem 5.14 (Fundamental Property) and the definition of $\boldsymbol{\mathcal{V}}[\![\mathbf{t}]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$. □

Lemma 5.17 (Free Theorem: Parametric Condition)
$\mathbf{D}, \boldsymbol{\alpha} :: * \vdash \boldsymbol{\rho_i}(\mathbf{t}_1) :: *$, $\mathbf{D} \vdash \boldsymbol{\rho_1}(\mathbf{t}_g) :: *$, $\mathbf{D} \vdash \boldsymbol{\rho_2}(\mathbf{t}_f) :: *$,
$\mathbf{D}; \mathbf{G} \vdash \mathbf{m} : \boldsymbol{\rho_i}(\forall \boldsymbol{\alpha}{::}{*}.\,(\mathbf{t}_1 \times (\mathbf{t}_2 \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha})$,
$\mathbf{D}; \mathbf{G} \vdash \mathbf{m}_f : \mathbf{t}_g \to \mathbf{t}_f$, $\mathbf{D}; \mathbf{G} \vdash \mathbf{m}_g : \mathbf{t}_2 \to \mathbf{t}_g$,
$(\mathbf{m}_a, \mathbf{m}'_a) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}_1]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}[\boldsymbol{\alpha} \mapsto (\mathbf{t}_g, \mathbf{t}_f, \mathbf{R})]}$, *then*
$(\mathbf{m}_f\ (\mathbf{m}\,[\mathbf{t}_g]\ \langle \mathbf{m}_a, \mathbf{m}_g \rangle), \mathbf{m}\,[\mathbf{t}_f]\ \langle \mathbf{m}'_a, \mathbf{m}_f \circ \mathbf{m}_g \rangle) \in \boldsymbol{\mathcal{E}}[\![\mathbf{t}_f]\!]^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\rho}}$

Lemma 5.18 (Free Theorem: $\eta^{\ell,\mathsf{s}}_{\mathbf{k}}$ shuffling)
Let $\boldsymbol{\rho} = [\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_{\zeta}$.

If $\boldsymbol{\Sigma} \vdash \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1} : \mathsf{T}_\ell\,\mathsf{s}^{\hat{+}}$ *then* $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \mathbf{m_1}[(\mathsf{T}_\ell\,\mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\,\mathsf{s}]\!], \eta^{\ell,\mathsf{s}}_{\mathbf{k}} \rangle) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\,\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$

Proof

Must show: $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \mathbf{m}[(\mathsf{T}_\ell\,\mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\,\mathsf{s}]\!], \eta^{\ell,\mathsf{s}}_{\mathbf{k}} \rangle) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\,\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$

Have: $\mathbf{m_1}[(\mathsf{T}_\ell\,\mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\,\mathsf{s}]\!], \eta^{\ell,\mathsf{s}}_{\mathbf{k}} \rangle \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_2}$ by Lemma 5.16 (Canonical Forms).

Must show: $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_2}) \in \boldsymbol{\mathcal{V}}[\![\mathsf{T}_\ell\,\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$

Consider arbitrary $\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_{\boldsymbol{\beta}}$ such that $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_{\boldsymbol{\beta}}) \in Rel^{\boldsymbol{\Sigma}}_{*}$.

Let $\boldsymbol{\rho}' = \boldsymbol{\rho}\,[\boldsymbol{\beta} \mapsto (\mathbf{t}_1, \mathbf{t}_2, \mathbf{R}_{\boldsymbol{\beta}})]$.

Must show: $(\mathbf{m_1}[\mathbf{t}_1/\boldsymbol{\beta}], \mathbf{m_2}[\mathbf{t}_2/\boldsymbol{\beta}]) \in \boldsymbol{\mathcal{E}}[\![((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\beta})) \to \boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$

Have: $\mathbf{m_1}[\mathbf{t}_1/\boldsymbol{\beta}] \longmapsto^* \boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \mathbf{t}_1) \times (\mathsf{s}^{\hat{+}} \to \mathbf{t}_1)).\mathbf{m_{11}}$ by Lemma 5.16 (Canonical Forms).

Have: $\mathbf{m_2}[\mathbf{t}_2/\boldsymbol{\beta}] \longmapsto^* \boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \mathbf{t}_2) \times (\mathsf{s}^{\hat{+}} \to \mathbf{t}_2)).\mathbf{m_{22}}$ by Lemma 5.16 (Canonical Forms).

Must show: $(\boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \mathbf{t}_1) \times (\mathsf{s}^{\hat{+}} \to \mathbf{t}_1)).\mathbf{m_{11}}, \boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \mathbf{t}_2) \times (\mathsf{s}^{\hat{+}} \to \mathbf{t}_2)).\mathbf{m_{22}})$
$\in \boldsymbol{\mathcal{V}}[\![((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})) \to \boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$

(1) Consider arbitrary $\mathbf{m}'_1, \mathbf{m}'_2$ such that $(\mathbf{m}'_1, \mathbf{m}'_2) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

Must show: $(\mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}], \mathbf{m_{22}}[\mathbf{m}'_2/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$

Assume that $\ell \not\sqsubseteq \zeta$:

Have: $(\mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}], \mathbf{m_{22}}[\mathbf{m}'_2/\mathbf{x}]) \in Atom\,[\boldsymbol{\beta}]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$ by assumption.

Have: $\mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}] \longmapsto^* \mathbf{m}_{1ir}$ and irred$(\mathbf{m}_{1ir})$ by Lemma 5.16 (Canonical Forms).

Have: $\mathbf{m_{22}}[\mathbf{m}'_2/\mathbf{x}] \longmapsto^* \mathbf{m}_{2ir}$ and irred$(\mathbf{m}_{2ir})$ by Lemma 5.16 (Canonical Forms).

Must show: $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \boldsymbol{\mathcal{V}}[\![\boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

Must show: $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathbf{R}_{\boldsymbol{\beta}}$

Have: $(\mathbf{prj}_1\ \mathbf{m}'_1, \mathbf{prj}_1\ \mathbf{m}'_2) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ from 1.

Have: $\ell \not\sqsubseteq \zeta \implies \mathbf{R}_{\boldsymbol{\beta}} = Atom\,[\mathbf{t}_1, \mathbf{t}_2]^{\boldsymbol{\Sigma}}$ by definition of $\boldsymbol{\rho}$ on $\boldsymbol{\alpha}_{\preceq}$ and previous fact.

Have: $\mathbf{R}_\beta = \mathit{Atom}\,[\mathbf{t}_1, \mathbf{t}_2]^{\boldsymbol{\Sigma}}$ by assumption $\ell \not\sqsubseteq \zeta$ and previous fact.

Have: $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathit{Atom}\,[\beta]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$ by Lemma 5.15 (Preservation).

Have: $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathit{Atom}\,[\mathbf{t}_1, \mathbf{t}_2]^{\boldsymbol{\Sigma}}$ by definition of $\boldsymbol{\rho}'$.

Therefore: $(\mathbf{m}_{1ir}, \mathbf{m}_{2ir}) \in \mathbf{R}_\beta$.

Assume that $\ell \sqsubseteq \zeta$:

(2) Have: $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}) \in \boldsymbol{\mathcal{V}}[\![\mathsf{T}_\ell\ \mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ by Theorem 5.14. (Fundamental Property)

Let $\mathbf{R} = \left\{ \begin{array}{l} (\mathbf{m_1}, \mathbf{m_2}) \in \mathit{Atom}\left[\mathbf{t}_1, \mathsf{T}_\ell\ \hat{\mathsf{s}^+}\right]^{\boldsymbol{\Sigma}} \mid \\ \qquad \ell \sqsubseteq \zeta \implies \exists \mathbf{m}.\mathbf{m_2} \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m} \wedge (\mathbf{m_1}, \mathbf{m}\,[\mathbf{t}_2]\,\mathbf{m}'_\mathbf{2}) \in \mathbf{R}_\beta \end{array} \right\}$.

Observe that: $\boldsymbol{\rho}$ on $\boldsymbol{\alpha}_\preceq$ requires that if $\ell \not\sqsubseteq \zeta$, then $\mathbf{R} = \mathit{Atom}\left[\mathbf{t}_1, \mathsf{T}_\ell\ \hat{\mathsf{s}^+}\right]^{\boldsymbol{\Sigma}}$, hence the $\ell \sqsubseteq \zeta$ condition.

Let $\boldsymbol{\rho}'' = \boldsymbol{\rho}\left[\boldsymbol{\beta} \mapsto (\mathbf{t}_1, \mathsf{T}_\ell\ \hat{\mathsf{s}^+}, \mathbf{R})\right]$.

Instantiating 2 with $(\mathbf{t}_1, \mathsf{T}_\ell\ \mathsf{s}^+, \mathbf{R})$:

$(\mathbf{m_1}[\mathbf{t}_1/\boldsymbol{\beta}], \mathbf{m_1}[\mathsf{T}_\ell\ \hat{\mathsf{s}^+}/\boldsymbol{\beta}]) \in \boldsymbol{\mathcal{E}}[\![((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})) \to \boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$

Now we will show: $(\mathbf{m}'_\mathbf{1}, \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \eta^{\ell,\mathsf{s}}_\mathbf{k}\rangle) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$

Recall: $(\mathbf{m}'_\mathbf{1}, \mathbf{m}'_\mathbf{2}) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

Therefore: $(\mathbf{prj}_1\ \mathbf{m}'_\mathbf{1}, \mathbf{prj}_1\ \mathbf{m}'_\mathbf{2}) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

Observe that: $(\mathbf{prj}_1\ \mathbf{m}'_\mathbf{1}, \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!]) \in \mathit{Atom}\left[\mathbf{t}_1, \mathsf{T}_\ell\ \hat{\mathsf{s}^+}\right]^{\boldsymbol{\Sigma}}$.

Have: $(\mathbf{prj}_1\ \mathbf{m}'_\mathbf{1}, \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!]) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$ by previous fact, definition of $\mathbf{R}$, and definition of $\boldsymbol{\rho}$ on $\boldsymbol{\alpha}_\preceq$.

Must show: $(\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1}, \eta^{\ell,\mathsf{s}}_\mathbf{k}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+ \to \boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$

Assume that $(\mathbf{m_{s1}}, \mathbf{m_{s2}}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$:

Must show: $((\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1})\ \mathbf{m_{s1}}, \eta^{\ell,\mathsf{s}}_\mathbf{k}\ \mathbf{m_{s2}}) \in \mathbf{R}$ Have: $\ell \sqsubseteq \zeta$ by assumption.

Must show: $\exists \mathbf{m}.\eta^{\ell,\mathsf{s}}_\mathbf{k}\ \mathbf{m_{s2}} \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m} \wedge ((\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1})\ \mathbf{m_{s1}}, \mathbf{m}\,[\mathbf{t}_2]\mathbf{m}'_\mathbf{2}) \in \mathbf{R}_\beta$

Observe that: $\eta^{\ell,\mathsf{s}}_\mathbf{k}\ \mathbf{m_{s2}} \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\beta}) \times \mathsf{s}^+ \to \boldsymbol{\beta}).((\mathbf{prj}_2\ \mathbf{x})\ \ \mathbf{m_{s2}})$.

Therefore: $\exists \mathbf{m} = (\boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\beta}) \times \hat{\mathsf{s}^+} \to \boldsymbol{\beta}).((\mathbf{prj}_2\ \mathbf{x})\ \mathbf{m_{s2}}))$.
$\eta^{\ell,\mathsf{s}}_\mathbf{k}\ \mathbf{m_{s2}} \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m}$.

Must show: $((\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1})\ \mathbf{m_{s1}}, \mathbf{m}\,[\mathbf{t}_2]\mathbf{m}'_\mathbf{2}) \in \mathbf{R}_\beta$

Observe that: $(\boldsymbol{\lambda}\mathbf{x}{:}((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\beta}) \times \hat{\mathsf{s}^+} \to \boldsymbol{\beta}).((\mathbf{prj}_2\ \mathbf{x})\ \mathbf{m_{s2}}))\,[\mathbf{t}_2]\ \mathbf{m}'_\mathbf{2} \longmapsto^* (\mathbf{prj}_2\ \mathbf{m}'_\mathbf{2})\ \mathbf{m_{s2}}$.

Must show: $((\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1})\ \mathbf{m_{s1}}, (\mathbf{prj}_2\ \mathbf{m}'_\mathbf{2})\ \mathbf{m_{s2}}) \in \mathbf{R}_\beta$

Recall: $(\mathbf{m}'_\mathbf{1}, \mathbf{m}'_\mathbf{2}) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^+ \to \boldsymbol{\beta})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$

Therefore: $(\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1}, \mathbf{prj}_2\ \mathbf{m}'_\mathbf{2}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+ \to \boldsymbol{\beta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

Have: $((\mathbf{prj}_2\ \mathbf{m}'_\mathbf{1})\ \mathbf{m_{s1}}, \mathbf{prj}_2\ \mathbf{m}'_\mathbf{2}\ \mathbf{m_{s2}}) \in \mathbf{R}_\beta$ by assumption that $(\mathbf{m_{s1}}, \mathbf{m_{s2}}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$.

Therefore: $(\mathbf{m_1}[\mathrm{t}_1/\beta]\ \mathbf{m}'_1, \mathbf{m_1}[\mathsf{T}_\ell\ \hat{\mathsf{s}^+}/\beta]\ \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle) \in \boldsymbol{\mathcal{E}}[\![\beta]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}''}$.

Recall: $\mathbf{m_1}[\mathsf{T}_\ell\ \hat{\mathsf{s}^+}/\beta]\ \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle \longmapsto^* \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_2}$ and $\mathbf{m_1}[\mathrm{t}_1/\beta]\ \mathbf{m}'_1 \longmapsto^* \mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}]$.

Have: $(\mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}], \mathbf{m_2}\ [\mathrm{t}_2]\ \mathbf{m}'_2) \in \mathbf{R}_{\boldsymbol{\beta}}$ by definition of $\mathbf{R}$.

Recall: $\mathbf{m_2}\ [\mathrm{t}_2]\ \mathbf{m}'_2 \longmapsto^* \mathbf{m_{22}}[\mathbf{m}'_2/\mathbf{x}]$.

Therefore: $(\mathbf{m_{11}}[\mathbf{m}'_1/\mathbf{x}], \mathbf{m_{22}}[\mathbf{m}'_2/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\beta]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}'}$.

□

5.4 Translation

Lemma 5.19
If $\ell \preceq \mathsf{s}$ *then* $\mathcal{L}^+_\ell; \mathcal{L}^+_\sqsubseteq, \preceq^+ \vdash \mathbf{pf}[\![\ell \preceq \mathsf{s}]\!] : (\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \mathsf{s}^+)$ *and* $\mathbf{D}_\ell; \mathbf{G}_\ell, \mathbf{G}_\preceq \vdash \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{s}]\!] : (\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \hat{\mathsf{s}^+})$

Lemma 5.20 (Translation is Complete)
If $\Gamma \vdash \mathsf{e} : \mathsf{s}$ *then* $\exists \mathbf{m}. \Gamma \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m}$

Theorem 5.21 (Translation Preserves Well-Typedness)
If $\Gamma \vdash \mathsf{e} : \mathsf{s}$ *and* $\Gamma \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m}$ *then* $\mathcal{L}^+_\ell; \mathcal{L}^+_\sqsubseteq, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathsf{s}^+$

Proof

Proof by induction on the translation derivation, $\Gamma \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m}$. □

Lemma 5.22 (Cross Lang. Relation Respects Target Relation)
Let $\boldsymbol{\rho} = [\![\mathcal{L}^+_\ell]\!]^{\boldsymbol{\Sigma}}_\zeta$ *and* $\boldsymbol{\delta} = \boldsymbol{\rho}_{\mathbf{i}}$ *(since* $\boldsymbol{\rho_1} = \boldsymbol{\rho_2}$*).*

1. *If* $(\mathsf{e}, \mathbf{m}) \in \mathcal{E}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$ *and* $(\mathbf{m}, \mathbf{m}') \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ *then* $(\mathsf{e}, \mathbf{m}') \in \mathcal{E}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$
2. *If* $(\mathsf{v}, \mathbf{u}) \in \mathcal{V}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$ *and* $(\mathbf{u}, \mathbf{u}') \in \boldsymbol{\mathcal{V}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ *then* $(\mathsf{v}, \mathbf{u}') \in \mathcal{V}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$

Proof

We prove 1 and 2 simultaneously. Proof by induction on the structure of the type s.

We give the cases for $\mathsf{s}_1 \to \mathsf{s}_2$ and $\mathsf{T}_\ell\ \mathsf{s}_1$.

Case *1*

Must show: $(\mathsf{e}, \mathbf{m}') \in \mathcal{E}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

By definition, it suffices to show $\mathsf{e} \longmapsto^* \mathsf{v}$, $\mathbf{m}' \longmapsto^* \mathbf{u}$ and $(\mathsf{v}, \mathbf{u}) \in \mathcal{V}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Have: $\mathsf{e} \longmapsto^* \mathsf{v}$ from $(\mathsf{e}, \mathbf{m}) \in \mathcal{E}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Have: $\mathbf{m}' \longmapsto^* \mathbf{u}'$ from $(\mathbf{m}, \mathbf{m}') \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

Have: $(\mathsf{v}, \mathbf{u}') \in \mathcal{V}^+_\zeta[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$ by *2*.

Case *2*, $\mathsf{s}_1 \to \mathsf{s}_2$

Must show: $(\lambda \mathsf{x} : \mathsf{s}_1.\, \mathsf{e}_2, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}^+_1.\mathbf{m}'_2) \in \mathcal{V}^+_\zeta[\![\mathsf{s}_1 \to \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$

Consider arbitrary $\mathsf{e}'_1, \mathbf{m}'_1$ such that $(\mathsf{e}'_1, \mathbf{m}'_1) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_1]\!]_{\boldsymbol{\delta}}$.

Must show: $(\mathsf{e}_2[\mathsf{e}'_1/\mathsf{x}], \mathbf{m}'_2[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_2]\!]_{\boldsymbol{\delta}}$

Instantiating $(\lambda \mathsf{x} : \mathsf{s}_1.\, \mathsf{e}_2, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}^+_1.\mathbf{m_2}) \in \mathcal{V}^+_\zeta[\![\mathsf{s}_1 \to \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$ with $(\mathsf{e}'_1, \mathbf{m}'_1) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_1]\!]_{\boldsymbol{\delta}}$:

Therefore: $(\mathsf{e}_2[\mathsf{e}'_1/\mathsf{x}], \mathbf{m_2}[\mathbf{m}'_1/\mathbf{x}]) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

Instantiating $(\lambda \mathsf{x} : \mathsf{s}_1.\, \mathsf{e}_2, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}^+_1.\mathbf{m_2}) \in \mathcal{V}^+_\zeta[\![\mathsf{s}_1 \to \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$ with $(\mathsf{e}'_1, \mathbf{m}'_1) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_1]\!]_{\boldsymbol{\delta}}$:

Therefore: $(\mathsf{e}_2[\mathsf{e}'_1/\mathsf{x}], \mathbf{m_2}[\mathbf{m_1}/\mathbf{x}]) \in \mathcal{E}^+_\zeta[\![\mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

Have: $(\mathbf{m_1'}, \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ by Theorem 5.14 (Fundamental Property).

Instantiating $(\boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2}, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2'}) \in \boldsymbol{\mathcal{V}}[\![\mathsf{s_1^+} \rightarrow \mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ with $(\mathbf{m_1'}, \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$:

Therefore: $(\mathbf{m_2}[\mathbf{m_1'}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Observe that: $(\mathsf{e_2}[\mathsf{e_1'}/\mathsf{x}], \mathbf{m_2}[\mathbf{m_1'}/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\![\mathsf{s_2}]\!]_{\boldsymbol{\delta}}$, and $(\mathbf{m_2}[\mathbf{m_1'}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Have: $(\mathsf{e_2}[\mathsf{e_1'}/\mathsf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\![\mathsf{s_2}]\!]_{\boldsymbol{\delta}}$ by induction via part *1*.

Case *2*, $\mathsf{T}_\ell\ \mathsf{s_1}$

Have: $(\eta_\ell\ \mathsf{e_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}}) \in \mathcal{V}_\zeta^+[\![\mathsf{T}_\ell\ \mathsf{s_1}]\!]_{\boldsymbol{\delta}}$.

Have: $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}'}) \in \boldsymbol{\mathcal{V}}[\![(\mathsf{T}_\ell\ \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Must show: $(\eta_\ell\ \mathsf{e_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}'}) \in \mathcal{V}_\zeta^+[\![\mathsf{T}_\ell\ \mathsf{s_1}]\!]_{\boldsymbol{\delta}}$

Must show: $\exists \mathbf{m_1'}.(\mathbf{m_{11}'}[(\mathsf{T}_\ell\ \mathsf{s_1})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e_1}, \mathbf{m_1'}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\delta}}$

Have: $\exists \mathbf{m_1}.(\mathbf{m_{11}}[(\mathsf{T}_\ell\ \mathsf{s_1})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e_1}, \mathbf{m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\delta}}$ from $(\eta_\ell\ \mathsf{e_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}}) \in \mathcal{V}_\zeta^+[\![\mathsf{T}_\ell\ \mathsf{s_1}]\!]_{\boldsymbol{\delta}}$.

Now we will show: $\exists \mathbf{m_1'}.(\mathbf{m_1}, \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$

> Take $\mathbf{m_1'}$ to be $\mathbf{m_1}$.
>
> Have: $(\mathbf{m_1}, \mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ by Theorem 5.14 (Parametricity).
>
> Must show: $(\mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$
>
> Let $\boldsymbol{\rho'} = (\mathsf{T}_\ell\ \mathsf{s}^{\hat{+}}, \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}}, \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}})$.
>
> Instantiating $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_{11}'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ with $(\mathsf{T}_\ell\ \mathsf{s}^{\hat{+}}, \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}}, \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}})$:
>
> $(\mathbf{m_{11}}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}], \mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]) \in \boldsymbol{\mathcal{E}}[\![((\boldsymbol{\alpha}_\preceq\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times \mathsf{s}^+ \rightarrow \boldsymbol{\beta}) \rightarrow \boldsymbol{\beta}]\!]_{\boldsymbol{\rho'}}^{\boldsymbol{\Sigma}}$
>
> Therefore: $(\mathbf{m_{11}}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle) \in \boldsymbol{\mathcal{E}}[\![\boldsymbol{\beta}]\!]_{\boldsymbol{\rho'}}^{\boldsymbol{\Sigma}}$.
>
> Therefore: $(\mathbf{m_{11}}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \ell^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.
>
> Have: $(\mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \mathbf{m_{11}}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \ell^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ by Lemma 5.7 (Symmetry).
>
> Have: $(\mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \ell^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ by Lemma 5.13 (Transitivity).

Have: $\exists \mathbf{m_1'}.(\mathbf{m_1}, \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathbf{m_{11}'}[\mathsf{T}_\ell\ \mathsf{s_1^{\hat{+}}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s_1}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Have: $(\mathsf{e_1}, \mathbf{m_1'}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\delta}}$ from the induction hypothesis (*1*) applied to $\mathsf{s_1}$.

□

Theorem 5.23 (Translation Preserves Semantics)
If $\Gamma \vdash \mathsf{e} : \mathsf{s}$ *and* $\Gamma \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m}$ *then* $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s}$

Proof

By induction on the typing derivation, $\Gamma \vdash \mathsf{e} : \mathsf{s}$. There are two interesting cases. Other cases follow easily by induction.

By the definition of the cross language logical relation, all cases require us to prove $\Gamma \vdash \mathsf{e} : \mathsf{s}$ and $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathsf{s}^+$. We omit these below, since $\Gamma \vdash \mathsf{e} : \mathsf{s}$ follows by assumption and $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathsf{s}^+$ follows by Theorem 5.21, Translation Preserves Well-Typedness.

Case DT-Prot

Have: $\Gamma \vdash \eta_\ell\, \mathsf{e} : \mathsf{T}_\ell\, \mathsf{s}$.

Have: $\Gamma \vdash \eta_\ell\, \mathsf{e} : \mathbf{\Lambda\beta{::}{*}.\lambda x{:}((\alpha_\preceq\ \ \alpha_\ell\ \ \beta) \times (s^+ \to \beta)).((prj_2\, x)\ \ m)} \rightsquigarrow \mathsf{T}_\ell\, \mathsf{s}$.

Must show: $\Gamma \vdash \eta_\ell\, \mathsf{e} \simeq \mathbf{\Lambda\beta{::}{*}.\lambda x{:}((\alpha_\preceq\ \ \alpha_\ell\ \ \beta) \times (s^+ \to \beta)).((prj_2\, x)\ \ m)} : \mathsf{T}_\ell\, \mathsf{s}$

Consider arbitrary $\boldsymbol{\delta}, \gamma, \boldsymbol{\gamma}, \boldsymbol{\gamma}_\sqsubseteq, \boldsymbol{\gamma}_\preceq$ such that $\boldsymbol{\delta} = \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_\preceq \mapsto \hat{\boldsymbol{\alpha}}_\preceq\}$, $\boldsymbol{\gamma}_\sqsubseteq = [\![\mathcal{L}_\sqsubseteq^+]\!]$, $\boldsymbol{\gamma}_\preceq = [\![\preceq^+]\!]$, and $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_{\boldsymbol{\delta}}$.

Let $\mathsf{e}_1 = \gamma(\mathsf{e})$.

Let $\mathbf{m_2} = \boldsymbol{\delta}(\boldsymbol{\gamma}_\sqsubseteq(\boldsymbol{\gamma}_\preceq(\boldsymbol{\gamma}(\mathbf{m}))))$.

Must show: $(\eta_\ell\, \mathsf{e}_1, \mathbf{\Lambda\beta{::}{*}.\lambda x{:}((\hat{\alpha}_\preceq\ \ \hat{\alpha}_\ell\ \ \beta) \times s^{\hat{+}} \to \beta).((prj_2\, x)\ \ m_2)}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$

Must show: $(\eta_\ell\, \mathsf{e}_1, \mathbf{\Lambda\beta{::}{*}.\lambda x{:}((\hat{\alpha}_\preceq\ \ \hat{\alpha}_\ell\ \ \beta) \times s^{\hat{+}} \to \beta).((prj_2\, x)\ \ m_2)}) \in \mathcal{V}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$

Let $\mathbf{t}_{pf} = (\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ (\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}})$.

Let $\boldsymbol{\rho} = [\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$.

Must show: $\exists \mathbf{m}_2'.(\mathbf{\lambda x{:}t}_{pf}.((\mathbf{prj}_2\, \mathbf{x})\, \mathbf{m_2})\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}_2') \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\, \mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e}_1, \mathbf{m}_2') \in \mathcal{V}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$

Take $\mathbf{m}_2'$ to be $\mathbf{m_2}$.

Observe that:

$\mathbf{\lambda x{:}t}_{pf}.((\mathbf{prj}_2\, \mathbf{x})\, \mathbf{m_2})\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle \longmapsto^* \mathbf{prj}_2\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle \mathbf{m_2} \longmapsto^* \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2}$.

Therefore: $(\mathbf{\lambda x{:}t}_{pf}.((\mathbf{prj}_2\, \mathbf{x})\, \mathbf{m_2})\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\, \mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Must show: $(\mathsf{e}_1, \mathbf{m_2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$

Have: $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s}$ by applying the induction hypothesis to s.

Instantiating $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s}$ with $\boldsymbol{\delta}$,γ,$\boldsymbol{\gamma}$,$\boldsymbol{\gamma}_\sqsubseteq$, and $\boldsymbol{\gamma}_\preceq$: $(\gamma(\mathsf{e}), \boldsymbol{\delta}\boldsymbol{\gamma}_\sqsubseteq\boldsymbol{\gamma}_\preceq\boldsymbol{\gamma}(\mathbf{m})) \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Recall the definition of e_1 and $\mathbf{m_2}$: $(\mathsf{e}_1, \mathbf{m_2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Case DT-Bind

Have: $\Gamma \vdash \mathsf{bind}\, \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e}' : \mathsf{s}'$.

Have: $\Gamma \vdash \mathsf{bind}\, \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e}' : \mathsf{s}' \rightsquigarrow \mathbf{m}\ [(\mathsf{s}')^+]\ \langle \mathbf{pf}[\![\ell \preceq \mathsf{s}']\!], (\mathbf{\lambda x{:}s^+.m'})\rangle$.

Must show: $\Gamma \vdash \mathsf{bind}\, \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e}' \simeq \mathbf{m}\ [(\mathsf{s}')^+]\ \langle \mathbf{pf}[\![\ell \preceq \mathsf{s}']\!], (\mathbf{\lambda x{:}s^+.m'})\rangle : \mathsf{s}'$

Consider arbitrary $\boldsymbol{\delta}$,γ,$\boldsymbol{\gamma}$,$\boldsymbol{\gamma}_\sqsubseteq$, and $\boldsymbol{\gamma}_\preceq$ such that $\boldsymbol{\delta} = \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_\preceq \mapsto \hat{\boldsymbol{\alpha}}_\preceq\}$, $\boldsymbol{\gamma}_\sqsubseteq = [\![\mathcal{L}_\sqsubseteq^+]\!]$, $\boldsymbol{\gamma}_\preceq = [\![\preceq^+]\!]$, and $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_{\boldsymbol{\delta}}$.,

Let $\mathsf{e}_1 = \gamma(\mathsf{e})$.

Let $\mathsf{e}_2 = \gamma(\mathsf{e}')$.

Let $\mathbf{m_1} = \boldsymbol{\delta}(\boldsymbol{\gamma}_\sqsubseteq(\boldsymbol{\gamma}_\preceq(\boldsymbol{\gamma}(\mathbf{m}))))$.

Let $\mathbf{m_2} = \boldsymbol{\delta}(\boldsymbol{\gamma}_\sqsubseteq(\boldsymbol{\gamma}_\preceq(\boldsymbol{\gamma}(\mathbf{m}'))))$.

Must show: $(\mathsf{bind}\, \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2, \mathbf{m_1}\ [(\mathsf{s}')^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{s}']\!], (\mathbf{\lambda x{:}s^{\hat{+}}.m_2})\rangle) \in \mathcal{E}_\zeta^+[\![\mathsf{s}_2]\!]_{\boldsymbol{\delta}}$

Let $\mathbf{m}_f = \mathbf{\lambda y{:}}(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}.\mathbf{y}\ [\mathsf{s}'^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{s}']\!], \mathbf{\lambda x{:}s^{\hat{+}}.m_2}\rangle : \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}} \to \mathsf{s}'^{\hat{+}}$.

Let $\mathbf{m}_g = \mathbf{\lambda y{:}s^{\hat{+}}.\Lambda\beta{::}{*}.\lambda x{:}((\hat{\alpha}_\preceq\ \ \hat{\alpha}_\ell\ \ \beta) \times s^{\hat{+}} \to \beta).((prj_2\, x)\ \ y)} : \mathsf{s}^{\hat{+}} \to \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}$.

Observe that: $\mathbf{m}_g = \eta_{\mathbf{k}}^{\ell,\mathsf{s}}$, and $(\mathbf{m}_f \circ \mathbf{m}_g, \mathbf{\lambda x{:}s^{\hat{+}}.m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^{\hat{+}} \to \mathsf{s}'^{\hat{+}}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Have: $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{T}_\ell\, \mathsf{s}$ from induction hypothesis applied to $\Gamma \vdash \mathsf{e} : \mathsf{T}_\ell\, \mathsf{s}$.

Instantiating $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{T}_\ell\, \mathsf{s}$ with $\boldsymbol{\delta}$,γ,$\boldsymbol{\gamma}$,$\boldsymbol{\gamma}_\sqsubseteq$, and $\boldsymbol{\gamma}_\preceq$:

Have: $(\mathsf{e}_1, \mathbf{m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Therefore: $\mathsf{e}_1 \longmapsto^* \eta_\ell\, \mathsf{e}_1'$, $\mathbf{m_1} \longmapsto^* \mathbf{\Lambda\beta{::}{*}.m_{11}}$, $(\eta_\ell\, \mathsf{e}_1', \mathbf{\Lambda\beta{::}{*}.m_{11}}) \in \mathcal{V}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$..

Let $\boldsymbol{\rho} = [\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$.

Therefore: $\exists \mathbf{m}_1'$. such that $(\mathbf{m_{11}}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m}_1') \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\, \mathsf{s})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e}_1', \mathbf{m}_1') \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Have: $(\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_g\rangle, \mathbf{m}_g\ \mathbf{m}_1') \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s})^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ by definition of $\mathbf{m}_g$ and reduction.

Therefore: $(\mathbf{m}_f\ (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_g\rangle), \mathbf{m}_f\ (\mathbf{m}_g\ \mathbf{m}_1'))$.

Have: $(\mathbf{m}_f\ (\mathbf{m}_g\ \mathbf{m}_1'), \mathbf{m}_f\ (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_g\rangle))$ by Lemma 5.7 (Symmetry).

Now we will show: $(\mathsf{bind}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2, \mathbf{m}_f\ (\mathbf{m}_g\ \mathbf{m}_1')) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$

> Have: $\mathsf{bind}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2 \longmapsto^* \mathsf{bind}\ \mathsf{x} = \eta_\ell\ \mathsf{e}_1'\ \mathsf{in}\ \mathsf{e}_2 \longmapsto^1 \mathsf{e}_2[\mathsf{e}_1'/\mathsf{x}]$.
>
> Have: $\mathbf{m}_f\ (\mathbf{m}_g\ \mathbf{m}_1') \longmapsto^* \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}^{\hat{+}}.\mathbf{m_2}\ \mathbf{m}_1' \longmapsto^1 \mathbf{m_2}[\mathbf{m}_1'/\mathbf{x}]$.
>
> Must show: $(\mathsf{e}_2[\mathsf{e}_1'/\mathsf{x}], \mathbf{m_2}[\mathbf{m}_1'/\mathbf{x}]) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$
>
> Have: $\Gamma, \mathsf{x} : \mathsf{s} \vdash \mathsf{e}' \simeq \mathbf{m}' : \mathsf{s}'$ from induction hypothesis applied to $\Gamma, \mathsf{x} : \mathsf{s} \vdash \mathsf{e}' : \mathsf{s}'$.
>
> Instantiating $\Gamma, \mathsf{x} : \mathsf{s} \vdash \mathsf{e}' \simeq \mathbf{m}' : \mathsf{s}'$ with $\boldsymbol{\delta}, \gamma\,[\mathsf{x} \mapsto \mathsf{e}_1'], \boldsymbol{\gamma}\,[\mathbf{x} \mapsto \mathbf{m}_1'], \boldsymbol{\gamma}_{\sqsubseteq}$, and $\boldsymbol{\gamma}_{\preceq}$:
>
> Recall definition of e_1 and $\mathbf{m_1}$.
>
> Have: $(\mathsf{e}_1[\mathsf{e}_1'/\mathsf{x}], \mathbf{m_1}[\mathbf{m}_1'/\mathbf{x}]) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$ by Compositionality.

Have: $(\mathsf{bind}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2, \mathbf{m}_f\ (\mathbf{m}_g\ \mathbf{m}_1')) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$.

Have: $(\mathsf{bind}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2, \mathbf{m}_f\ (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_g\rangle)) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$ by Lemma 5.22 (Cross Lang. Relation Respects Target Relation).

Have: $(\mathbf{m}_f\ (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_g\rangle), (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \mathbf{m}_f \circ \mathbf{m}_g\rangle)) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}'^+]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ by Lemma 5.17 (Parametric Condition).

Have: $(\mathsf{bind}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2, (\mathbf{m_1}\ [(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{s}']\!], \mathbf{m}_f \circ \mathbf{m}_g\rangle)) \in \mathcal{E}^+_\zeta[\![\mathsf{s}']\!]_{\boldsymbol{\delta}}$ by Lemma 5.22 (Cross Lang. Relation Respects Target Relation).

□

5.5 Back-translation

Lemma 5.24
If $\Sigma \vdash \mathbf{m} : (\hat{\alpha}_{\preceq}\ \ \hat{\alpha}_{\ell}\ \ \mathsf{s}^{\hat{+}})$ *then* $\ell \preceq \mathsf{s}$.

Lemma 5.25 (Monotonicity)
If $(\mathbf{W}, \mathbf{m}) \in \mathbf{R}$, $(\mathbf{t}, \mathbf{R}) \in Rel^{\Uparrow}{}^{\Sigma}_{*}$, *and* $\mathbf{W}' \supseteq \mathbf{W}$, *then* $(\mathbf{W}', \mathbf{m}) \in \mathbf{R}$.

Proof

Follows by definition of $Rel^{\Uparrow}{}^{\Sigma}_{*}$. □

Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$)
If $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$, *and* $\mathbf{W}' \supseteq \mathbf{W}$ *then* $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$

Proof

Case $\exists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$.

For arbitrary $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\emptyset}$, $\mathbf{W}' \supseteq \mathbf{W}$, we must show that $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\emptyset}$ and $\exists \mathsf{e}.\Sigma; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$.

By definition of $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\emptyset}$, we know $\exists \mathsf{e}.\Sigma; \mathbf{W}_{\mathbf{k}}; \mathbf{W}_{\Gamma} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$.
The extra variables of $\mathbf{W}'$ make no difference in back-translation since they are not used in $\mathbf{m}$. Since $\mathbf{m}$ is back-translatable under $\mathbf{W}$, it is also back-translatable under $\mathbf{W}'$.
That is, the same back-translation derivation gives us $\Sigma; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$.

Case $\nexists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$.

For arbitrary $\mathbf{W}' \supseteq \mathbf{W}$, we must show $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

For arbitrary $\mathbf{W_1} \supseteq \mathbf{W}'$ and $(\mathbf{W_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, it suffices to show that $(\mathbf{W_1}, \mathbf{E}^{\not+}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, which follows from instantiating $(\mathbf{W}', \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$ with $\mathbf{W_1} \supseteq \mathbf{W}$ and $(\mathbf{W_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

□

Lemma 5.27 (Monotonicity of $\mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$)
If $(\mathbf{W}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, *and* $\mathbf{W}' \supseteq \mathbf{W}$, *then* $(\mathbf{W}', \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Proof

For arbitrary $\mathbf{W}' \supseteq \mathbf{W}$, we must show $(\mathbf{W}', \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

For arbitrary $\mathbf{W_1} \supseteq \mathbf{W}'$, $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not+}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Clearly $\mathbf{W_1} \supseteq \mathbf{W}$, so we instantiate the hypothesis $(\mathbf{W}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$ with $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.
Thus $(\mathbf{W_1}, \mathbf{E}^{\not+}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$. □

Lemma 5.28 (Monotonicity of $\mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$)
If $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$, *and* $\mathbf{W}' \supseteq \mathbf{W}$ *then* $(\mathbf{W}', \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$

Proof

Follows simply from Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$). □

Lemma 5.29
If $\boldsymbol{\Delta} \vdash \mathbf{t} :: \boldsymbol{\kappa}$, $\delta \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\Sigma}$, $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{W_k}, \mathbf{W}_{\Gamma} \vdash \mathbf{E}^{\not+} : \delta(\mathbf{t}) \Rightarrow \mathsf{s}'^{\hat{+}}$, *and* $(\forall\ \mathbf{u}, \mathbf{W}'.\mathbf{W}' \supseteq \mathbf{W} \wedge \Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{W}'_{\mathbf{k}}, \mathbf{W}'_{\Gamma} \vdash \mathbf{u} : \delta(\mathbf{t}) \implies \exists \mathsf{e}.\ \Sigma; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{E}^{\not+}[\mathbf{u}] : \mathsf{s}'^{\hat{+}} \Uparrow \mathsf{e})$ *then* $(\mathbf{W}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Proof

To show $(\mathbf{W}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ it suffices to show for arbitrary $\mathbf{W}' \supseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, that $(\mathbf{W}', \mathbf{E}^{\not\uparrow}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By the definition of $\mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $\exists \mathsf{e}.\boldsymbol{\Sigma}; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{m}] : \mathsf{s}'^{\hat{+}} \Uparrow \mathsf{e}$, which follows by instantiating $(\forall\, \mathbf{u}, \mathbf{W}'.\mathbf{W}' \supseteq \mathbf{W} \wedge \boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{W}'_{\mathbf{k}}, \mathbf{W}'_{\Gamma} \vdash \mathbf{u} : \boldsymbol{\delta}(\mathbf{t}) \implies \exists \mathsf{e}.\ \boldsymbol{\Sigma}; \mathbf{W}'_{\mathbf{k}}; \mathbf{W}'_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{u}] : \mathsf{s}'^{\hat{+}} \Uparrow \mathsf{e})$ with $\mathbf{W}' \supseteq \mathbf{W}$, and $\mathbf{m}$. □

Lemma 5.30 (Relation equivalence is transitive)
Let $(\mathbf{t}, \mathbf{R_1}) \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$, $(\mathbf{t}, \mathbf{R}) \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$, and $(\mathbf{t}, \mathbf{R}) \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$.
If $\mathbf{R_1} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \mathbf{R}$ and $\mathbf{R} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \mathbf{R_2}$ then $\mathbf{R_1} \equiv^{\mathbf{D};\mathbf{G}}_{\boldsymbol{\kappa}} \mathbf{R_2}$.

Proof

By induction on $\boldsymbol{\kappa}$.

Case $\boldsymbol{\kappa} = *$ We must show $\mathbf{R_1} \equiv^{\boldsymbol{\Sigma}}_{*} \mathbf{x_2}$.

- For arbitrary $(\mathbf{W_1}, \mathbf{m_1}) \in \mathbf{R_1}$, we must show $(\mathbf{W_1}, \mathbf{m_1}) \in \mathbf{R_2}$.
  From $\mathbf{R_1} \equiv^{\boldsymbol{\Sigma}}_{*} \mathbf{R}$, we know $(\mathbf{W_1}, \mathbf{m}) \in \mathbf{R}$.
  From $\mathbf{R} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \mathbf{R_2}$, we know $(\mathbf{W_1}, \mathbf{m_1}) \in \mathbf{R_2}$.
- For arbitrary $(\mathbf{W_1}, \mathbf{m_1}) \in \mathbf{R_2}$, we must show $(\mathbf{W_1}, \mathbf{m_1}) \in \mathbf{R_1}$.
  This case is exactly symmetric to the previous case

Case $\boldsymbol{\kappa} = \boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}$.

For arbitrary $(\mathbf{t}', \mathbf{R}') \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa_1}}$, we must show $\mathbf{R_1}\ (\mathbf{t}', \mathbf{R}') \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa_2}} \mathbf{R_2}\ (\mathbf{t}', \mathbf{R}')$.
From $\mathbf{R_1} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa_1} \rightarrow \boldsymbol{\kappa_2}} \mathbf{R}$, we know $\mathbf{R_1}\ (\mathbf{t}', \mathbf{R}') \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa_2}} \mathbf{R}\ (\mathbf{t}', \mathbf{R}')$.
From $\mathbf{R} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \mathbf{R_2}$, we know $\mathbf{R}\ (\mathbf{t}', \mathbf{R}') \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa_2}} \mathbf{R_2}\ (\mathbf{t}', \mathbf{R}')$.
The result follows by IH applied to $\boldsymbol{\kappa_2}$.

□

Lemma 5.31 (Back-translation type interpretation is well-formed)
If $\boldsymbol{\Delta} \vdash \mathbf{t} :: \boldsymbol{\kappa}$ and $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$ then

1. $(\boldsymbol{\delta}(\mathbf{t}), \mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}) \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$
2. *If $\boldsymbol{\delta}' \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$ such that $\boldsymbol{\delta} \equiv^{\boldsymbol{\Sigma}} \boldsymbol{\delta}'$, then $\mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}} \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}'}$*

Proof

By induction on the derivation $\boldsymbol{\Delta} \vdash \mathbf{t} :: \boldsymbol{\kappa}$.

$$\text{(FK-Var)}\ \frac{\boldsymbol{\alpha} :: \boldsymbol{\kappa} \in \boldsymbol{\Delta}}{\boldsymbol{\Delta} \vdash \boldsymbol{\alpha} :: \boldsymbol{\kappa}}$$

1. We must show $\boldsymbol{\delta}_{\mathbf{R}}(\boldsymbol{\alpha}) \in Rel^{\Uparrow}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$.
   Follows by assumption that $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$.
2. We must show $\boldsymbol{\delta}_{\mathbf{R}}(\boldsymbol{\alpha}) \equiv^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}} \boldsymbol{\delta}'_{\mathbf{R}}(\boldsymbol{\alpha})$.
   Follows by assumption that $\boldsymbol{\delta} \equiv^{\boldsymbol{\Sigma}} \boldsymbol{\delta}'$.

$$\text{(FK-Unit)}\ \frac{}{\boldsymbol{\Delta} \vdash \mathbf{1} :: *}$$

1. We must show $\mathcal{E}^{\Uparrow}[\![\mathbf{1}]\!]^{\Sigma}_{\delta} \in Rel^{\Uparrow}{}^{\Sigma}_{*}$.
   Follows by definition of $\mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
2. We must show $\mathcal{E}^{\Uparrow}[\![\mathbf{1}]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![\mathbf{1}]\!]^{\Sigma}_{\delta'}$, which follows trivially since back-translation does not rely on δ.

$$\text{(FK-Pair)}\ \frac{\Delta \vdash \mathbf{t}_1 :: * \qquad \Delta \vdash \mathbf{t}_2 :: *}{\Delta \vdash \mathbf{t}_1 \times \mathbf{t}_2 :: *}$$

Case $\exists \mathsf{s}_1, \mathsf{s}_2 . \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$, and $\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\Sigma}_{\delta} \in Rel^{\Uparrow}{}^{\Sigma}_{*}$.
   Follows by definition of $\mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
2. We must show $\mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\Sigma}_{\delta'}$, which follows trivially since back-translation does not rely on δ.

This subcase follows by exactly the same reasoning in all kind $*$ cases of proof, but it is repeated in detail anyway for completeness.

Case $\exists \mathsf{s}_1 . \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$, but $\nexists \mathsf{s}_2 . \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta} \in Rel^{\Uparrow}{}^{\Sigma}_{*}$.
   - For arbitrary $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$, we must show that $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, which follows from Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$), and
   - For arbitrary $\mathbf{E}^{\not\Uparrow}$, $\mathbf{t}$, suppose $(\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W}'_1)_{\mathbf{k}}, (\mathbf{W}'_1)_{\Gamma} \vdash \mathbf{E}^{\not\Uparrow} : \delta(\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2) \Rightarrow \mathsf{s}'^{\hat{+}}$, and
     $\forall \mathbf{W}'_2, \mathbf{u} . \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \land \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W}'_2)_{\mathbf{k}}, (\mathbf{W}'_2)_{\Gamma} \vdash \mathbf{u} : \mathbf{t} \implies .\exists \mathsf{e} . \mathbf{\Sigma}; (\mathbf{W}'_2)_{\mathbf{k}}; (\mathbf{W}'_2)_{\Gamma} \vdash \mathbf{E}^{\not\Uparrow}[\mathbf{u}] : \mathsf{s}'^{\hat{+}} \Uparrow \mathsf{e}$.
     We must show $\exists \mathsf{e}' . \mathbf{\Sigma}; (\mathbf{W}'_1)_{\mathbf{k}}; (\mathbf{W}'_1)_{\Gamma} \vdash \mathbf{E}^{\not\Uparrow}[\mathbf{m}] : \mathsf{s}'^{\hat{+}} \Uparrow \mathsf{e}'$.
     Instantiating $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$ with $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$, it suffices to show $(\mathbf{W}'_1, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, which follows from Lemma 5.29.

   This subcase follows by exactly the same reasoning in all kind $*$ cases of proof, but it is repeated in detail anyway for completeness.
2. We must show $\mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta'}$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta'}$.
     For arbitrary $(\mathbf{W_1}, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\Uparrow}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times [\mathbf{t}_2]]\!]^{\Sigma}_{\delta}$ and the definition of $\mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
     For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m}') \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not\Uparrow}[\mathbf{m}']) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
     Since δ' is irrelevant to back-translation, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\Uparrow}[\mathbf{m}']) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By Lemma 5.27 applied to $(\mathbf{W_1}, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$ we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By $(\mathbf{W_2}, \mathbf{E}^{\not\Uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta'}$ it suffices to show $(\mathbf{W_2}, \mathbf{m}') \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta'}$.
     Since $(\mathbf{W_2}, \mathbf{m}') \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, it suffices to show $\mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\Sigma}_{\delta'}$ and $\mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta'}$, which follow from the IH part 2 applied to $\Delta \vdash \mathsf{s}_1^{\hat{+}} :: *$ and $\Delta \vdash \mathbf{t}_2 :: *$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta'}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\Sigma}_{\delta}$.
     This direction is exactly symmetric to the previous direction.

Case $\exists s_2.t_2 = s_2^{\hat{+}}$, but $\nexists s_1.t_1 = s_1^{\hat{+}}$.
This case is symmetric to the previous case.

Case $\nexists s_1.t_1 = s_1^{\hat{+}}$ and $\nexists s_2.t_2 = s_2^{\hat{+}}$.
This case is identical to the non-translation subcases of the previous two cases.

$$(\text{FK-Sum})\ \frac{\mathbf{\Delta} \vdash \mathbf{t_1} :: * \qquad \mathbf{\Delta} \vdash \mathbf{t_2} :: *}{\mathbf{\Delta} \vdash \mathbf{t_1} + \mathbf{t_2} :: *}$$

Case $\exists s_1, s_2.t_1 = s_1^{\hat{+}}$, and $t_2 = s_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + s_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \in Rel^{\Uparrow}\ {}^{\mathbf{\Sigma}}_{*}$.
   Follows by definition of $\mathcal{E}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.
2. We must show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + s_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + s_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$, which follows trivially since back-translation does not rely on δ.

Case $\exists s_1.t_1 = s_1^{\hat{+}}$, but $\nexists s_2.t_2 = s_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta} \in Rel^{\Uparrow}\ {}^{\mathbf{\Sigma}}_{*}$.
   - For arbitrary $(\mathbf{W'_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$, $\mathbf{W'_2} \sqsupseteq \mathbf{W'_1}$, we must show that $(\mathbf{W'_2}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$, which follows from Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![t]\!]^{\mathbf{\Sigma}}_{\delta}$), and
   - For arbitrary $\mathbf{E}^{\not+}$, $\mathbf{t}$, suppose $(\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W'_1})_{\mathbf{k}}, (\mathbf{W'_1})_{\Gamma} \vdash \mathbf{E}^{\not+} : \delta(s_1^{\hat{+}} + t_2) \Rightarrow s'^{\hat{+}}$, and
     $\forall \mathbf{W'_2}, \mathbf{u}. \mathbf{W'_2} \sqsupseteq \mathbf{W'_1} \wedge \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W'_2})_{\mathbf{k}}, (\mathbf{W'_2})_{\Gamma} \vdash \mathbf{u} : \mathbf{t} \implies .\exists e.\mathbf{\Sigma}; (\mathbf{W'_2})_{\mathbf{k}}; (\mathbf{W'_2})_{\Gamma} \vdash \mathbf{E}^{\not+}[\mathbf{u}] : s'^{\hat{+}} \Uparrow e$.
     We must show $\exists e'.\mathbf{\Sigma}; (\mathbf{W'_1})_{\mathbf{k}}; (\mathbf{W'_1})_{\Gamma} \vdash \mathbf{E}^{\not+}[\mathbf{m}] : s'^{\hat{+}} \Uparrow e'$.
     Instantiating $(\mathbf{W'_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$ with $\mathbf{W'_1} \sqsupseteq \mathbf{W'_1}$, it suffices to show $(\mathbf{W'_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$, which follows from Lemma 5.29.
2. We must show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta'}$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     For arbitrary $(\mathbf{W_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not+}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$ and the definition of $\mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.
     For arbitrary $\mathbf{W_2} \sqsupseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not+}[\mathbf{m'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.
     From $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$, we know $\mathbf{m'} = \mathbf{inj_i}\ \mathbf{m''}$ and $(\mathbf{W_2}, \mathbf{m''}) \in \mathcal{E}^{\Uparrow}[\![t_i]\!]^{\mathbf{\Sigma}}_{\delta}$.
     Since δ' is irrelevant to back-translation, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not+}[\mathbf{m'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By Lemma 5.27 applied to $(\mathbf{W_1}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$ we conclude $(\mathbf{W_2}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By $(\mathbf{W_2}, \mathbf{E}^{\not+}) \in \mathcal{K}^{\Uparrow}[\![s_1^{\hat{+}} + t_2, s^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$ it suffices to show $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     Since $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$, it suffices to show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$ or $\mathcal{E}^{\Uparrow}[\![t_2]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\Uparrow}[\![t_2]\!]^{\mathbf{\Sigma}}_{\delta'}$, which follow from the IH part 2 applied to $\mathbf{\Delta} \vdash s_1^{\hat{+}} :: *$ and $\mathbf{\Delta} \vdash \mathbf{t_2} :: *$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta'}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} + t_2]\!]^{\mathbf{\Sigma}}_{\delta}$.
     This direction is exactly symmetric to the previous direction.

Case $\exists s_2.t_2 = s_2^{\hat{+}}$, but $\nexists s_1.t_1 = s_1^{\hat{+}}$.
This case is symmetric to the previous case.

Case $\nexists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$ and $\nexists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

This case is identical to the non-translation subcases of the previous two cases.

$$\text{(FK-Abs)}\ \frac{\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa} \vdash \mathbf{t} :: *}{\mathbf{\Delta} \vdash \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t} :: *}$$

Case $\exists \mathsf{s}. \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t} = \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}$.

1. We must show $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \in Rel^{\hat{\uparrow}}\ {}^{\mathbf{\Sigma}}_{*}$.

   Follows by definition of $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.

2. We must show $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$, which follows trivially since back-translation does not rely on δ.

Case $\nexists \mathsf{s}. \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t} = \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}}$.

1. We must show $\mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta} \in Rel^{\hat{\uparrow}}\ {}^{\mathbf{\Sigma}}_{*}$.

   - For arbitrary $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \mathbf{t}{::}\boldsymbol{\alpha}.\,]\!]^{\mathbf{\Sigma}}_{\delta}$, $\mathbf{W}'_2 \supseteq \mathbf{W}'_1$, we must show that $(\mathbf{W}'_2, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$, which follows from Lemma 5.26 (Monotonicity of $\mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$), and
   - For arbitrary $\mathbf{E}^{\not\uparrow}$, $\mathbf{t}$, suppose $(\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W}'_1)_{\mathbf{k}}, (\mathbf{W}'_1)_{\Gamma} \vdash \mathbf{E}^{\not\uparrow} : \delta(\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}) \Rightarrow \mathsf{s}'^{\hat{+}}$, and
     $\forall \mathbf{W}'_2, \mathbf{u}. \mathbf{W}'_2 \supseteq \mathbf{W}'_1 \wedge \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W}'_2)_{\mathbf{k}}, (\mathbf{W}'_2)_{\Gamma} \vdash \mathbf{u} : \mathbf{t} \implies .\exists \mathsf{e}. \mathbf{\Sigma}; (\mathbf{W}'_2)_{\mathbf{k}}; (\mathbf{W}'_2)_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{u}] : \mathsf{s}'^{\hat{+}} \uparrow \mathsf{e}$.
     We must show $\exists \mathsf{e}'. \mathbf{\Sigma}; (\mathbf{W}'_1)_{\mathbf{k}}; (\mathbf{W}'_1)_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{m}] : \mathsf{s}'^{\hat{+}} \uparrow \mathsf{e}'$.
     Instantiating $(\mathbf{W}'_1, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$ with $\mathbf{W}'_1 \supseteq \mathbf{W}'_1$, it suffices to show $(\mathbf{W}'_1, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$, which follows from Lemma 5.29.

2. We must show $\mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta'}$.

   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     For arbitrary $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[\mathbf{m}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$ and the definition of $\mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.
     For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m}') \in \mathcal{V}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.
     Since δ' is irrelevant to back-translation, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By Lemma 5.27 applied to $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$ we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     By $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta'}$ it suffices to show $(\mathbf{W_2}, \mathbf{m}') \in \mathcal{V}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta'}$.
     For arbitrary $(\mathbf{t}', \mathbf{R}) \in Rel^{\hat{\uparrow}}\ {}^{\mathbf{\Sigma}}_{\boldsymbol{\kappa}}$, it suffices to show $\mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta[\boldsymbol{\alpha}\mapsto(\mathbf{t}',\mathbf{R})]} \equiv^{\mathbf{\Sigma}}_{*} \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta[\boldsymbol{\alpha}\mapsto(\mathbf{t}',\mathbf{R})]}$ which follows from the IH part 2 applied to $\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa} \vdash \mathbf{t} :: *$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}\delta'$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}]\!]^{\mathbf{\Sigma}}_{\delta}$.
     This direction is exactly symmetric to the previous direction.

$$\text{(FK-Arrow)}\ \frac{\mathbf{\Delta} \vdash \mathbf{t}_1 :: * \qquad \mathbf{\Delta} \vdash \mathbf{t}_2 :: *}{\mathbf{\Delta} \vdash \mathbf{t}_1 \rightarrow \mathbf{t}_2 :: *}$$

Case $\exists \mathsf{s}_1, \mathsf{s}_2. \mathbf{t}_1 \rightarrow \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} \rightarrow \mathsf{s}_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \rightarrow \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta} \in Rel^{\hat{\uparrow}}\ {}^{\mathbf{\Sigma}}_{*}$.

   Follows by definition of $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\delta}$.

2. We must show $\mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} \to s_2^{\hat{+}}]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} \to s_2^{\hat{+}}]\!]^{\Sigma}_{\delta'}$, which follows trivially since back-translation does not rely on δ.

Case $\nexists s_1, s_2.t_1 \to t_2 = s_1^{\hat{+}} \to s_2^{\hat{+}}$.

1. We must show $\mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta} \in \mathit{Rel}^{\Uparrow}{}^{\Sigma}_{*}$.
   - For arbitrary $(\mathbf{W_1'}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$, $\mathbf{W_2'} \supseteq \mathbf{W_1'}$, we must show that $(\mathbf{W_2'}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$, which follows from Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![t]\!]^{\Sigma}_{\delta}$), and
   - For arbitrary $\mathbf{E}^{\not\uparrow}$, $\mathbf{t}$, suppose $(\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W_1'})_{\mathbf{k}}, (\mathbf{W_1'})_{\Gamma} \vdash \mathbf{E}^{\not\uparrow} : \delta(t_1 \to t_2) \Rightarrow s'^{\hat{+}}$, and
     $\forall \mathbf{W_2'}, \mathbf{u}.\mathbf{W_2'} \supseteq \mathbf{W_1'} \wedge \mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, (\mathbf{W_2'})_{\mathbf{k}}, (\mathbf{W_2'})_{\Gamma} \vdash \mathbf{u} : \mathbf{t} \implies .\exists e.\mathbf{\Sigma}; (\mathbf{W_2'})_{\mathbf{k}}; (\mathbf{W_2'})_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{u}] : s'^{\hat{+}} \Uparrow e$.
     We must show $\exists e'.\mathbf{\Sigma}; (\mathbf{W_1'})_{\mathbf{k}}; (\mathbf{W_1'})_{\Gamma} \vdash \mathbf{E}^{\not\uparrow}[\mathbf{m}] : s'^{\hat{+}} \Uparrow e'$.
     Instantiating $(\mathbf{W_1'}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$ with $\mathbf{W_1'} \supseteq \mathbf{W_1'}$, it suffices to show $(\mathbf{W_1'}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta}$, which follows from Lemma 5.29.
2. We must show $\mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta'}$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta'}$.
     For arbitrary $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[\mathbf{m}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$ and the definition of $\mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
     For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
     Since δ' is irrelevant to back-translation, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By Lemma 5.27 applied to $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$ we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$.
     By $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![t_1 \to t_2, s^{\hat{+}}]\!]^{\Sigma}_{\delta'}$ it suffices to show $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta'}$.
     Since $(\mathbf{W_2}, \mathbf{m'}) \in \mathcal{V}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta}$, it suffices to show $\mathcal{E}^{\Uparrow}[\![t_1]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![t_1]\!]^{\Sigma}_{\delta'}$ and $\mathcal{E}^{\Uparrow}[\![t_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{*} \mathcal{E}^{\Uparrow}[\![t_2]\!]^{\Sigma}_{\delta'}$, which follow from the IH part 2 applied to $\mathbf{\Delta} \vdash t_1 :: *$ and $\mathbf{\Delta} \vdash t_2 :: *$.
   - If $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 \to t_2]\!]^{\Sigma}_{\delta'}$ then $(\mathbf{W_1}, \mathbf{m}) \in \mathcal{E}^{\Uparrow}[\![t_1 + t_2]\!]^{\Sigma}_{\delta}$.
     This direction is exactly symmetric to the previous direction.

$$\text{(FK-Fun)}\ \frac{\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa_1} \vdash \mathbf{t} :: \boldsymbol{\kappa_2}}{\mathbf{\Delta} \vdash \boldsymbol{\lambda\alpha{::}\kappa}.\mathbf{t} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}}$$

1. We must show $(\delta(\boldsymbol{\lambda\alpha{::}\kappa}.\mathbf{t}), \mathcal{T}^{\Uparrow}[\![\boldsymbol{\lambda\alpha{::}\kappa}.\mathbf{t} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta}) \in \mathit{Rel}^{\Uparrow}{}^{\Sigma}_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}}$.
   For arbitrary $(\mathbf{t_1}, \mathbf{R_1}) \in \mathit{Rel}^{\Uparrow}{}^{\Sigma}_{\boldsymbol{\kappa_1}}$, we must show
   (a) $(\mathbf{t}[\mathbf{t_1}/\boldsymbol{\alpha}], \mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta[\boldsymbol{\alpha} \mapsto (\mathbf{t_1}, \mathbf{R_1})]}) \in \mathit{Rel}^{\Uparrow}{}^{\Sigma}_{\boldsymbol{\kappa_2}}$ which follows from IH part 1 applied to $\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa_1} \vdash \mathbf{t} :: \boldsymbol{\kappa_2}$, and
   (b) For arbitrary $(\mathbf{t_1'}, \mathbf{R_1'}) \in \mathit{Rel}^{\Uparrow}{}^{\Sigma}_{\boldsymbol{\kappa_1}}$, if $(\mathbf{t_1}, \mathbf{R_1}) \equiv^{\Sigma}_{\boldsymbol{\kappa_1}} (\mathbf{t_1'}, \mathbf{R_1'})$ then $\mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta[\boldsymbol{\alpha} \mapsto (\mathbf{t_1}, \mathbf{R_1})]} \equiv^{\Sigma}_{\boldsymbol{\kappa_2}} \mathcal{T}^{\Uparrow}[\![\mathbf{t} :: \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta[\boldsymbol{\alpha} \mapsto (\mathbf{t_1'}, \mathbf{R_1'})]}$, which follows from IH part 2 applied to $\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa_1} \vdash \mathbf{t} :: \boldsymbol{\kappa_2}$.
2. We must show $\mathcal{T}^{\Uparrow}[\![\boldsymbol{\lambda\alpha{::}\kappa}.\mathbf{t} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}} \mathcal{T}^{\Uparrow}[\![\boldsymbol{\lambda\alpha{::}\kappa}.\mathbf{t} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2}]\!]^{\Sigma}_{\delta'}$
   This follows from IH part 2 applied to $\mathbf{\Delta}, \boldsymbol{\alpha} :: \boldsymbol{\kappa_1} \vdash \mathbf{t} :: \boldsymbol{\kappa_2}$.

$$\text{(FK-App)}\ \frac{\mathbf{\Delta} \vdash \mathbf{t_1} :: \boldsymbol{\kappa_1} \to \boldsymbol{\kappa_2} \qquad \mathbf{\Delta} \vdash \mathbf{t_2} :: \boldsymbol{\kappa_1}}{\mathbf{\Delta} \vdash \mathbf{t_1}\ \mathbf{t_2} :: \boldsymbol{\kappa_2}}$$

1. We must show $(\delta(t_1\, t_2), \mathcal{T}^{\Uparrow}[\![t_1\, t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta}) \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa}$.

   By IH part 1 applied to $\Delta \vdash t_1 :: \kappa_1 \to \kappa_2$ we know $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1 \to \kappa_2}$. By definition of $Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1 \to \kappa_2}$ it suffices to show $(\delta(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta}) \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1}$, which follows by IH part 1 applied to $\Delta \vdash t_2 :: \kappa_1$.

2. We must show $\mathcal{T}^{\Uparrow}[\![t_1\, t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa} \mathcal{T}^{\Uparrow}[\![t_1\, t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}$

   It suffices to show

   $(\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta}\, (\delta(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta})) \equiv^{\Sigma}_{\kappa_2} (\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}\, (\delta'(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta'}))$.

   By IH part 2 we know $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa_1 \to \kappa_2} \mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}$ and

   $\mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa_1} \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}$,

   By IH part 1 we know $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1 \to \kappa_2}$ and $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'} \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1 \to \kappa_2}$.

   Instantiating $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa_1 \to \kappa_2} \mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}$ with $\mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta}$ we know

   $(\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta}\, (\delta(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta})) \equiv^{\Sigma}_{\kappa_2} (\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}\, (\delta(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta}))$.

   Instantiating $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'} \in Rel^{\Uparrow}\,^{\Sigma}_{\kappa_1 \to \kappa_2}$ with $\mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa_1} \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}$, we know

   $(\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}\, (\delta(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta})) \equiv^{\Sigma}_{\kappa_2} (\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1 \to \kappa_2]\!]^{\Sigma}_{\delta'}\, (\delta'(t_2), \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_1]\!]^{\Sigma}_{\delta'}))$.

   The result follows by Lemma 5.30 (Relation equivalence is transitive).

□

Lemma 5.32 (Back-translation compositionality)
If $\Delta, \alpha :: \kappa_2 \vdash t_1 :: \kappa_2$, $\Delta \vdash t_2 :: \kappa_2$, *and* $R_2 = \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa_2]\!]^{\Sigma}_{\delta}$, *then* $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa_1]\!]^{\Sigma}_{\delta[\alpha \mapsto (t_2, R_2)]} \equiv^{\Sigma}_{\kappa_1} \mathcal{T}^{\Uparrow}[\![t_1[\alpha/t_2] :: \kappa_1]\!]^{\Sigma}_{\delta}$.

Theorem 5.33 (Coherence for back-translation relation)
If $\Delta \vdash t_1 :: \kappa$, $\Delta \vdash t_2 :: \kappa$, $t_1 \equiv t_2$, *and* $\delta \in \mathcal{D}^{\Uparrow}[\![\Delta]\!]^{\Sigma}$, *then* $\mathcal{T}^{\Uparrow}[\![t_1 :: \kappa]\!]^{\Sigma}_{\delta} \equiv^{\Sigma}_{\kappa} \mathcal{T}^{\Uparrow}[\![t_2 :: \kappa]\!]^{\Sigma}_{\delta}$.

Lemma 5.34 (Fundamental Property of Back-translation Logical Relation)
If $D_\ell, \Delta; G_\ell, G_\preceq, G_k, \Gamma^{\hat{+}}, G_k, \Gamma \vdash m : t$ *then* $D_\ell, \Delta; G_\ell, G_\preceq, G_k, \Gamma^{\hat{+}}, G_k, \Gamma \vDash^{\Uparrow} m : t$

Proof idea: Proof by induction on the typing derivation $D_\ell, \Delta; G_\ell, G_\preceq, G_k, \Gamma^{\hat{+}}, G_k, \Gamma \vdash m : t$. Most cases will have subcases for either $\exists s.t = s^{\hat{+}}$ or not.

For brevity, W for $(G_k; \Gamma'^{\hat{+}})$ unless the world requries particular attention.

Proof

Case FT-Unit

We know

$$\text{(FT-Unit)}\ \frac{}{D_\ell, \Delta; G_\ell, G_\preceq, G_k, \Gamma^{\hat{+}}, \Gamma \vdash \langle\rangle : 1}$$

We must show $D_\ell, \Delta; G_\ell, G_\preceq, G_k, \Gamma^{\hat{+}}, \Gamma \vDash^{\Uparrow} \langle\rangle : 1$.

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \gamma$. such that $\delta \in \mathcal{D}^{\Uparrow}[\![\Delta]\!]^{\Sigma}$, $((G_k; \Gamma_2^{\hat{+}}), \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$, $((G_k; \Gamma_2^{\hat{+}}), \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$, we must show

$((G_k; \Gamma_2^{\hat{+}}), \langle\rangle) \in \mathcal{E}^{\Uparrow}[\![1]\!]^{\Sigma}_{\delta}$, which follows by FD-Unit.

Case FT-Var

We know

$$\frac{\mathbf{x} : \mathbf{t} \in \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{G_k}, \mathbf{\Gamma}}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{x} : \mathbf{t}}$$

We must show $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vDash^{\hat{\uparrow}} \mathbf{x} : \mathbf{t}$.

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \boldsymbol{\delta}, \gamma, \boldsymbol{\gamma}$. such that $\boldsymbol{\delta} \in \mathcal{D}^{\hat{\uparrow}}[\![\mathbf{\Delta}]\!]^{\mathbf{\Sigma}}$, $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, we must show

$\boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{x}))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$.

For brevity, let $\mathbf{m}' = \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{x})))$.

Case $\mathbf{x} : \mathbf{t} \in \mathbf{\Gamma}$

Then we know $\nexists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$, and $\mathbf{x}$ is in the domain of $\boldsymbol{\gamma}$. Let $\boldsymbol{\gamma}(\mathbf{x}) = \mathbf{m}$. We must show $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$

By definition of $\mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$ and by assumption that $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, we know $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$

Case $\mathbf{x} : \mathbf{t} \in \Gamma_1^{\hat{+}}$

Then we know $\exists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$ and $\mathbf{x}$ is in the domain of γ. Let $\gamma(\mathbf{x}) = \mathbf{m}$. We must show $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$.

By definition of $\mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$ and by assumption that $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, we know $(\mathbf{W}, \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\mathbf{x} : \mathbf{t} \in \mathbf{G_k}$

Then we know $\mathbf{x} = \mathbf{k}$, $\mathbf{t} = \mathsf{s}^{\hat{+}} \rightarrow (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}$ and $\mathbf{x}$ is not in the domain of $\boldsymbol{\gamma}$ or γ.

We must show $((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \mathbf{k}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}} \rightarrow (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, which follows from FD-K.

Case $\mathbf{x} : \mathbf{t} \in \Gamma_2^{\hat{+}}$

Then we know $\mathbf{x} : \mathbf{t} \in \Gamma_2^{\hat{+}}$, and $\mathbf{t} = \mathsf{s}^{\hat{+}}$ and $\mathbf{x}$ is not in the domain of $\boldsymbol{\gamma}$ or γ.

$((\mathbf{G_k}; \Gamma_2^{\hat{+}}), \mathbf{x}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$ follows by FD-Var.

Case $\mathbf{x} : \mathbf{t} \in \mathbf{G}_\ell$

Then we know $\exists \ell.\mathbf{t} = \hat{\boldsymbol{\alpha}}_\ell$ and $\mathbf{x}$ is not in the domain of $\boldsymbol{\gamma}$ or γ. We must show $(\mathbf{W}, \mathbf{x}) \in \mathcal{E}^{\hat{\uparrow}}[\![\hat{\boldsymbol{\alpha}}_\ell]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$.

For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{E}^{\not{+}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\hat{\boldsymbol{\alpha}}_\ell, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \mathbf{E}^{\not{+}}[\mathbf{x}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$. There is only one $\mathbf{E}^{\not{+}}$ in which a $\mathbf{x} : \hat{\boldsymbol{\alpha}}_\ell$ can appear: $[\cdot]_{\mathbf{T}}$. However, the result type cannot be $\mathsf{s}^{\hat{+}}$. So, since there is no well-typed context of type $(\hat{\boldsymbol{\alpha}}_\ell) \Rightarrow \mathsf{s}^{\hat{+}}$, the statement $\forall (\mathbf{W}', \mathbf{E}^{\not{+}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\hat{\boldsymbol{\alpha}}_\ell, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}.(\mathbf{W}', \mathbf{E}^{\not{+}}[\mathbf{x}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$ holds vacuously. Therefore $(\mathbf{W}', \mathbf{x}) \in \mathcal{E}^{\hat{\uparrow}}[\![\hat{\boldsymbol{\alpha}}_\ell]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$

Note the implication here is that $\mathbf{x}$ can appear in a term, but never in evaluation position. It can be, for instance, passed as an argument to a function that never its argument other than by passing it to another function that behaves similarly. The back-translation reduction will cause such a term to be discarded. So $\mathbf{x}$ exists only as a proof object to allow terms corresponding to protected source terms to be well-typed.

Case $\mathbf{x} : \mathbf{t} \in \mathbf{G}_{\preceq}$

The argument for this case is similar to the previous case. We know $\mathbf{x}$ is one of the proof constructors $\hat{\mathbf{p}}_{\mathbf{1}}$, $\hat{\mathbf{p}}_{\times}$, $\hat{\mathbf{p}}_{\rightarrow}$, $\hat{\mathbf{p}}_{\mathbf{T_1}}$, or $\hat{\mathbf{p}}_{\mathbf{T_2}}$.

Formally, we know $\nexists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$ and $\mathbf{x}$ is not in the domain of $\boldsymbol{\gamma}$ or γ. We must show $(\mathbf{W}, \mathbf{x}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$.

For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{E}^{\not{+}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}', \mathbf{E}^{\not{+}}[\mathbf{x}]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_{\boldsymbol{\delta}}$. However, by inspecting the types of $\mathbf{x}$ and of each $\mathbf{E}^{\not{+}}$, we know no such $\mathbf{E}^{\not{+}}$ exists.

Consider for instance $\mathbf{x} = \hat{\mathbf{p}}_{\times}$. We know:
$\mathbf{t} = \forall\boldsymbol{\beta}_{\ell}{::}{*}.\ \forall\boldsymbol{\alpha_1}{::}{*}.\ \forall\boldsymbol{\alpha_2}{::}{*}.\ ((\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha_1}) \times (\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \boldsymbol{\beta}_{\ell}\ \ \boldsymbol{\alpha_2})) \to (\hat{\boldsymbol{\alpha}}_{\preceq}\ \ \boldsymbol{\beta}_{\ell}\ \ (\boldsymbol{\alpha_1} \times \boldsymbol{\alpha_2}))$
A term of this type can appear in an evaluation context such as $[\cdot]_{\mathbf{T}}\,[\hat{\boldsymbol{\alpha}}_{\ell}]\,[\mathbf{t}_1]\,[\mathbf{t}_2]\,\mathbf{m}'$. However, the result of this evaluation context is not of translation type, and cannot appear in any other $\mathbf{E}^{\not\dagger}$ in evaluation position. For every possible $\mathbf{x} : \mathbf{t} \in \mathbf{G}_{\preceq}$, the same is true of all $\mathbf{E}^{\not\dagger}$ in which $\mathbf{x}$ can appear.

Since there is no $\mathbf{E}^{\not\dagger}$ that results in a translation type, $(\mathbf{W}', \mathbf{x}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ is vacuously true.

Case FT-Pair

We know

$$\frac{\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \qquad \mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_2}{\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \langle \mathbf{m_1}, \mathbf{m_2} \rangle : \mathbf{t}_1 \times \mathbf{t}_2}$$

We must show $\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vDash^{\Uparrow} \langle \mathbf{m_1}, \mathbf{m_2} \rangle : \mathbf{t}_1 \times \mathbf{t}_2$.

For arbitrary $\Gamma^{\hat{+}}_{\mathbf{1}}, \Gamma^{\hat{+}}_{\mathbf{2}}, \boldsymbol{\delta}, \gamma, \boldsymbol{\gamma}$. such that $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\boldsymbol{\Gamma}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show
$(\mathbf{W}, \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\langle \mathbf{m_1}, \mathbf{m_2} \rangle)))) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For brevity, let $\mathbf{m}'_{\mathbf{1}} = \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{m_1})))$, and $\mathbf{m}'_{\mathbf{2}} = \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{m_2})))$.

Case $\exists \mathsf{s}_1, \mathsf{s}_2. \mathbf{t}_1 \times \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}$

We must show $(\mathbf{W}, \langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By the IH applied to $\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1$, and $\mathbf{D}_{\ell}, \boldsymbol{\Delta}; \mathbf{G}_{\ell}, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_2$, we know:
$(\mathbf{W}, \mathbf{m}'_{\mathbf{1}}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ and $(\mathbf{W}, \mathbf{m}'_{\mathbf{2}}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By FD-Pair, we know $(\mathbf{W}, \langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\exists \mathsf{s}_1. \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$, but $\nexists \mathsf{s}_2. \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

We must show $(\mathbf{W}, \langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}', \mathbf{E}^{\not\dagger}[\langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By the definition of $\mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \langle \mathbf{m_1}, \mathbf{m_2} \rangle) \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By the definition of $\mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \mathbf{m}'_{\mathbf{1}}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ and $(\mathbf{W}', \mathbf{m}'_{\mathbf{2}}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follow by the IH applied to $\boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{m_i} : \mathbf{t}_i$ and Lemma 5.26.

Case $\exists \mathsf{s}_2. \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$, but $\nexists \mathsf{s}_1. \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$.

The proof is analogous to the previous case.

Case $\nexists \mathsf{s}_1. \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$ and $\nexists \mathsf{s}_2. \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

The proof is analogous to the two previous cases. We repeat it since both types are non-translation types and it is not obviously symmetric.
We must show $(\mathbf{W}, \langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \mathbf{E}^{\not\dagger}[\langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By definition of $\mathcal{K}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \langle \mathbf{m}'_{\mathbf{1}}, \mathbf{m}'_{\mathbf{2}} \rangle) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By definition of $\mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ it suffices to show $(\mathbf{W}', \mathbf{m}'_{\mathbf{i}}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_i]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follows by the IH applied to $\boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{m_i} : \mathbf{t}_i$ and Lemma 5.26.

Case FT-Prj

We know

$$\frac{\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2}{\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{prj_i}\ \mathbf{m} : \mathbf{t}_i}$$

We must show $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vDash^{\hat{\uparrow}} \mathbf{prj_i}\ \mathbf{m} : \mathbf{t}_i$.

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \boldsymbol{\delta}, \gamma, \boldsymbol{\gamma}$. such that $\boldsymbol{\delta} \in \mathcal{D}^{\hat{\uparrow}}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\boldsymbol{\Gamma}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show

$(\mathbf{W}, \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{prj_i}\ \mathbf{m})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_i]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

For brevity, let $\mathbf{m}' = \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$.

We consider only the case when $\mathbf{i} = 1$ and $\mathbf{t}_i = \mathbf{t}_1$, as the other case follows symmetrically.

Case $\exists \mathsf{s}_1, \mathsf{s}_2 . \mathbf{t}_1 \times \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}$

We must show $(\mathbf{W}, \mathbf{prj}_1\ \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By FD-Subterm and FD-Prj, it suffices to show $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follows by the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$.

Case $\exists \mathsf{s}_1 . \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$, but $\nexists \mathsf{s}_2 . \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

We must show $(\mathbf{W}, \mathbf{prj}_1\ \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$, we know $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By definition of $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}, \mathbf{prj}_1\ [\cdot]_{\mathbf{T}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2, \boldsymbol{\delta}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \langle \mathbf{m}_1', \mathbf{m}_2' \rangle) \in \mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \mathbf{prj}_1 \langle \mathbf{m}_1', \mathbf{m}_2' \rangle) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Note that $\mathbf{prj}_1\ \langle \mathbf{m}_1', \mathbf{m}_2' \rangle = \mathbf{E}^{\not{f}}[\mathbf{u}]$.

By FD-Val and the definition of $\mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}', \mathbf{m}_1') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follows by assumption that $(\mathbf{W}', \langle \mathbf{m}_1', \mathbf{m}_2' \rangle) \in \mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \times \mathbf{t}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\exists \mathsf{s}_2 . \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$, but $\nexists \mathsf{s}_1 . \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$.

We must show $(\mathbf{W}, \mathbf{prj}_1\ \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

For arbitrary $\mathbf{W_1} \sqsupseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not{f}}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}_1, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not{f}}[\mathbf{prj}_1\ \mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 \times \mathbf{t}_2$, we know $(\mathbf{W_1}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

By $(\mathbf{W_1}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, to show $(\mathbf{W_1}, \mathbf{E}^{\not{f}}[\mathbf{prj}_1\ \mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not{f}}[\mathbf{prj}_1\ [\cdot]_{\mathbf{T}}]) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Note that $\mathbf{prj}_1\ [\cdot]_{\mathbf{T}} : (\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}) \Rightarrow \mathbf{t}_1$, so $\mathbf{E}^{\not{f}}[\mathbf{prj}_1\ [\cdot]_{\mathbf{T}}] = \mathbf{E}_1^{\not{f}}$.

For arbitrary $\mathbf{W_2} \sqsupseteq \mathbf{W_1}$, $(\mathbf{W_2}, \langle \mathbf{m}_1', \mathbf{m}_2' \rangle) \in \mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not{f}}[\mathbf{prj}_1 \langle \mathbf{m}_1', \mathbf{m}_2' \rangle]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Note that $\mathbf{E}^{\not{f}}[\mathbf{prj}_1\ \langle \mathbf{m}_1', \mathbf{m}_2' \rangle] = \mathbf{E}_1^{\not{f}}[\mathbf{u}]$, so by FD-Val, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not{f}}[\mathbf{m}_1']) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

From $(\mathbf{W_2}, \langle \mathbf{m}_1', \mathbf{m}_2' \rangle) \in \mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ we know $(\mathbf{W_2}, \mathbf{m}_1') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$. Instantiating the latter with $\mathbf{E}^{\not{f}}$, we conclude $(\mathbf{W_2}, \mathbf{E}^{\not{f}}[\mathbf{m}_1']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\nexists \mathsf{s}_1 . \mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$ and $\nexists \mathsf{s}_2 . \mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

This case is analogous to the previous case. Note in the previous case, the fact that $\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$ had no effect, since the type of the pair $\mathbf{t}_1 \times \mathsf{s}_2^{\hat{+}}$ and the type of the projection $\mathbf{t}_1$ are not translation types.

Case FT-Sum

We know

$$\frac{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_i}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{inj_i\ m} : \mathbf{t}_1 + \mathbf{t}_2}$$

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \boldsymbol{\gamma}$. such that $\delta \in \mathcal{D}^{\hat{\uparrow}}[\![\mathbf{\Delta}]\!]^{\mathbf{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_\delta$, we must show

$(\mathbf{W}, \delta(\gamma(\boldsymbol{\gamma}(\mathbf{inj_i\ m})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.

For brevity, let $\mathbf{m}' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$.

Case $\exists \mathsf{s}_1, \mathsf{s}_2.\mathbf{t}_1 + \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}$.

We must show $(\mathbf{W}, \mathbf{inj_i\ m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By FD-Sum, it suffices to show $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_i^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, which follows by the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1$.

Case $\exists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$, but $\nexists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$.

We must show $(\mathbf{W}, \mathbf{inj_i\ m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.

For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \mathbf{E}^{\not\downarrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, we must show $(\mathbf{W}', \mathbf{E}^{\not\downarrow}[\mathbf{inj_i\ m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By $(\mathbf{W}', \mathbf{E}^{\not\downarrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show $(\mathbf{W}', \mathbf{inj_i\ m}') \in \mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.

By definition of $\mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show $(\mathbf{W}', \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_i]\!]^{\mathbf{\Sigma}}_\delta$, which follows by IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_i$ and Lemma 5.26.

Case $\exists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$, but $\nexists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$.

This case is exactly identical to the previous case. Note that in the previous case, we never use the fact that $\mathbf{t}_i'$ may be a translation type.

Case $\nexists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$, and $\nexists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$.

This case is exactly identical to the previous two cases. Note that in the previous cases, we never use the fact that $\mathbf{t}_i'$ may be a translation type.

Case FT-Case

We know

$$\frac{\begin{array}{c}\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2 \\ \mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathbf{t}_1, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t} \qquad \mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_2} : \mathbf{t}_2, \mathbf{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}\end{array}}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{case\,m\,of\,inj_1\ x_1.\,m_1 \,\|\, inj_2\ x_2.\ m_2} : \mathbf{t}}$$

We must show $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vDash^{\hat{\uparrow}} \mathbf{case\,m\,of\,inj_1\ x_1.\,m_1 \,\|\, inj_2\ x_2.\ m_2} : \mathbf{t}$

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \boldsymbol{\gamma}$. such that $\delta \in \mathcal{D}^{\hat{\uparrow}}[\![\mathbf{\Delta}]\!]^{\mathbf{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$,

$(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_\delta$, we must show

$(\mathbf{W}, \delta(\gamma(\boldsymbol{\gamma}(\mathbf{case\,m\,of\,inj_1\ x_1.\,m_1 \,\|\, inj_2\ x_2.\ m_2})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}]\!]^{\mathbf{\Sigma}}_\delta$.

For brevity, $\mathbf{m}' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$, $\mathbf{m}_1' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m_1})))$, and $\mathbf{m}_2' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m_2})))$.

Case $\exists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$

Case $\exists \mathsf{s}_1^{\hat{+}}, \mathsf{s}_2^{\hat{+}}.\mathbf{t}_1 + \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}$

We must show $(\mathbf{W}, \mathbf{case\,m'\,of\,inj_1\ x_1.\,m_1' \,\|\, inj_2\ x_2.\ x_2'}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

Let $\mathbf{W'} = (\mathbf{G_k}; \Gamma^{\hat{+}}_{\mathbf{2}}, \mathbf{x} : \mathsf{s}^{\hat{+}}_1)$.

By Lemma 5.28 (Monotonicity of $\mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$), we know $(\mathbf{W'}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\Sigma}_{\delta}$ and $(\mathbf{W'}, \boldsymbol{\gamma}) \in \mathcal{E}^{\Uparrow}[\![\boldsymbol{\Gamma}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}}_1, \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}$, we know $(\mathbf{W'}, \mathbf{m'_1}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}}_2, \boldsymbol{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}$, we know $(\mathbf{W'}, \mathbf{m'_2}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Subterm and FD-Case, we conclude $(\mathbf{W}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\, \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\, \mathbf{m'_2}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$

Case $\exists \mathsf{s}^{\hat{+}}_1 . \mathbf{t}_1 = \mathsf{s}^{\hat{+}}_1$, but $\nexists \mathsf{s}^{\hat{+}}_2 . \mathbf{t}_2 = \mathsf{s}^{\hat{+}}_2$.

We must show $(\mathbf{W}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\, \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\, \mathbf{x'_2}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By $(\mathbf{W}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W}, \mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\, \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\, \mathbf{m'_2}) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Note that $\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\, \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\, \mathbf{m'_2} : \mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2 \Rightarrow \mathsf{s}^{\hat{+}}$ is an $\mathbf{E}^{\not\ell}$.

For arbitrary $\mathbf{W'} \sqsupseteq \mathbf{W}$, $(\mathbf{W'}, \mathbf{inj_i}\, \mathbf{m''}) \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show

$(\mathbf{W'}, \mathbf{case}\, \mathbf{inj_i}\, \mathbf{m''}\, \mathbf{of}\, \mathbf{inj}_1\, \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\, \mathbf{m'_2}) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Val and definition of $\mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W'}, \mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\mathbf{i} = 1$

From $(\mathbf{W'}, \mathbf{inj_i}\, \mathbf{m''}) \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$ we know $(\mathbf{W'}, \mathbf{m''}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}^{\hat{+}}_1, \boldsymbol{\Gamma} \vdash \mathbf{m_1} : \mathsf{s}^{\hat{+}}$, we know $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}^{\hat{+}}_1, \boldsymbol{\Gamma} \vDash^{\Uparrow} \mathbf{m_1} : \mathsf{s}^{\hat{+}}$.

Recall that $\mathbf{W'} \sqsupseteq \mathbf{W}$. By Lemma 5.28 (Monotonicity of $\mathcal{G}^{\Uparrow}[\![\Gamma]\!]^{\Sigma}_{\delta}$), we know $(\mathbf{W'}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\Sigma}_{\delta}$ and $(\mathbf{W'}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\boldsymbol{\Gamma}]\!]^{\Sigma}_{\delta}$.

We instantiate $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}^{\hat{+}}_1, \boldsymbol{\Gamma} \vDash^{\Uparrow} \mathbf{m_1} : \mathsf{s}^{\hat{+}}$. with $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\Sigma}$, $(\mathbf{W'}, \gamma\, [\mathbf{x_1} \mapsto \mathbf{m''}]) \in \mathcal{G}^{\Uparrow}[\![\Gamma^{\hat{+}}_{\mathbf{1}}, \mathbf{x_1} : \mathsf{s}^{\hat{+}}_1]\!]^{\Sigma}_{\delta}$, $(\mathbf{W'}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\boldsymbol{\Gamma}]\!]^{\Sigma}_{\delta}$ and conclude:

$(\mathbf{W'}, \boldsymbol{\delta}(\gamma\, [\mathbf{x_1} \mapsto \mathbf{m''}]\, (\boldsymbol{\gamma}(\mathbf{m_1})))) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Note that this is the same as $(\mathbf{W'}, \mathbf{m'_1}[\mathbf{m''}/\mathbf{x_1}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\mathbf{i} = 2$

From $(\mathbf{W'}, \mathbf{inj_i}\, \mathbf{m''}) \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$ we know $(\mathbf{W'}, \mathbf{m''}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma}, \mathbf{x_2} : \mathbf{t}_2 \vdash \mathbf{m_2} : \mathsf{s}^{\hat{+}}$, we know $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma}, \mathbf{x_2} : \mathbf{t}_2 \vDash^{\Uparrow} \mathbf{m_2} : \mathsf{s}^{\hat{+}}$.

We instantiate this with $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\boldsymbol{\Delta}]\!]^{\Sigma}$, $(\mathbf{W'}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\Sigma}_{\delta}$, $(\mathbf{W'}, \boldsymbol{\gamma}\, [\mathbf{x_2} \mapsto \mathbf{m''}]) \in \mathcal{G}^{\Uparrow}[\![\boldsymbol{\Gamma}, \mathbf{x_2} : \mathbf{t}_2]\!]^{\Sigma}_{\delta}$ and conclude:

$(\mathbf{W'}, \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}\, [\mathbf{x_2} \mapsto \mathbf{m''}]\, (\mathbf{m_2})))) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Note that this is exactly $(\mathbf{W'}, \mathbf{m'_2}[\mathbf{m''}/\mathbf{x_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\exists \mathsf{s}^{\hat{+}}_2 . \mathbf{t}_2 = \mathsf{s}^{\hat{+}}_2$, but $\nexists \mathsf{s}^{\hat{+}}_1 . \mathbf{t}_1 = \mathsf{s}^{\hat{+}}_1$.

The proof of this case is symmetric to the previous case. The primary difference is the cases for $\mathbf{i} = 1$ and $\mathbf{i} = 2$ are switched.

Case $\nexists \mathsf{s}^{\hat{+}}_1, \mathsf{s}^{\hat{+}}_2 . \mathbf{t}_1 = \mathsf{s}^{\hat{+}}_1$ or $\mathbf{t}_2 = \mathsf{s}^{\hat{+}}_2$

This case is analogous to the previous cases. We repeat it since both types are non-translation types and it is not obviously symmetric.

We must show $(\mathbf{W}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By $(\mathbf{W}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W}, \mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\ \mathbf{m'_2}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$

For arbitrary $\mathbf{W'} \supseteq \mathbf{W}$, $(\mathbf{W'}, \mathbf{inj_i}\ \mathbf{m''}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W'}, \mathbf{case}\, \mathbf{inj_i}\ \mathbf{m''}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Val and the definition of $\mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W'}, \mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Recall $\mathbf{W'} \supseteq \mathbf{W}$, so by Lemma 5.26 (Monotonicity of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$), and the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma}, \mathbf{x_i} : \mathbf{t}_i \vdash \mathbf{m_i} : \mathbf{t}$, we know $(\mathbf{W'}, \boldsymbol{\gamma}\, [\mathbf{x_i} \mapsto \mathbf{m''}]\, (\mathbf{m'_i})) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Note that this is exactly $(\mathbf{W'}, \mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\nexists \mathsf{s}.\mathbf{t} = \mathsf{s}^{\hat{+}}$

We must show $(\mathbf{W}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

For arbitrary $\mathbf{W_1} \supseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not\lightning}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\lightning}[\mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}]) \in \mathcal{O}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\exists \mathsf{s}_1, \mathsf{s}_1.\mathbf{t}_1 + \mathbf{t}_2 = \mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}$

Note that in this case $\mathbf{E}^{\not\lightning}[\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}] \neq \mathbf{E}_1^{\not\lightning}$, since $\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2} : \mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}} \Rightarrow \mathbf{t}$. Therefore FD-Val cannot apply.

By FD-Stuck, it suffices to show $(\mathbf{W_1}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{E}^{\not\lightning}[\mathbf{m'_1}] \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{E}^{\not\lightning}[\mathbf{m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$. (We ignore alpha-renaming w.l.o.g.)

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W_1}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} + \mathsf{s}_2^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Let $\mathbf{W_2} = (\mathbf{W_1}, \mathbf{x_i} : \mathbf{t}_i)$

By Lemma 5.28 (Monotonicity of $\mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]^{\Sigma}_{\delta}$), we know $(\mathbf{W_2}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$ and $(\mathbf{W_2}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma}, \mathbf{x_i} : \mathbf{t}_i \vdash \mathbf{m_i} : \mathbf{t}$, we know $(\mathbf{W_2}, \mathbf{m'_i}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

By Lemma 5.27 (Monotonicity of $\mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$), we know $(\mathbf{W_2}, \mathbf{E}^{\not\lightning}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Instantiating $(\mathbf{W_2}, \mathbf{m'_i}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$ with $(\mathbf{W_2}, \mathbf{E}^{\not\lightning})$, we know $(\mathbf{W_2}, \mathbf{E}^{\not\lightning}[\mathbf{m'_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Subterm and FD-Case, $(\mathbf{W_1}, \mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{E}^{\not\lightning}[\mathbf{m'_1}] \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{E}^{\not\lightning}[\mathbf{m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\exists \mathsf{s}_1^{\hat{+}}.\mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$, but $\nexists \mathsf{s}_2^{\hat{+}}.\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$.

We must show $(\mathbf{W_1}, \mathbf{E}^{\not\lightning}[\mathbf{case}\, \mathbf{m'}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\ \mathbf{x_2}.\ \mathbf{m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m'} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W_1}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By $(\mathbf{W_1}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\lightning}[\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\ \mathbf{m'_2}]) \in \mathcal{K}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Note that $\mathbf{E}^{\not\lightning}[\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\ \mathbf{m'_2}] = \mathbf{E}_1^{\not\lightning}$ since $\mathbf{case}\, [\cdot]_{\mathbf{T}}\, \mathbf{of}\, \mathbf{inj}_1\ \mathbf{x_1}.\, \mathbf{m'_1} \parallel \mathbf{inj}_2\, \mathbf{x_2}.\ \mathbf{m'_2} : \mathsf{s}_1^{\hat{+}} + \mathbf{t}_2 \Rightarrow \mathbf{t}$.

For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{inj_i}\ \mathbf{m''}) \in \mathcal{V}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}} + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show

$(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{case\,inj_i\;m''\,of\,inj_1\;x_1.\,m'_1 \parallel inj_2\;x_2.\;m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Val, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\mathbf{i} = 1$

Note that $(\mathbf{W_2}, \mathbf{m''}) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$ and $\mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}_1^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}$, we know $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}_1^{\hat{+}}, \mathbf{\Gamma} \vDash^{\Uparrow} \mathbf{m_1} : \mathbf{t}$.

Recall that $\mathbf{W_2} \supseteq \mathbf{W}$. So by Lemma 5.28 (Monotonicity of $\mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]^{\Sigma}_{\delta}$), we know $(\mathbf{W_2}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]^{\Sigma}_{\delta}$, $(\mathbf{W_2}, \gamma\,[\mathbf{x_1} \mapsto \mathbf{m''}]) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}, \mathbf{x_1} : \mathsf{s}_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

We instantiate $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_1} : \mathsf{s}_1^{\hat{+}}, \mathbf{\Gamma} \vDash^{\Uparrow} \mathbf{m_1} : \mathbf{t}$. with $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\mathbf{\Delta}]\!]^{\Sigma}$, $(\mathbf{W_2}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]^{\Sigma}_{\delta}$, $(\mathbf{W_2}, \gamma\,[\mathbf{x_1} \mapsto \mathbf{m''}]) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}, \mathbf{x_1} : \mathsf{s}_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Therefore we know $(\mathbf{W_2}, \gamma\,[\mathbf{x_1} \mapsto \mathbf{m''}]\,(\mathbf{m'_1})) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$. Note that this is exactly $(\mathbf{W_2}, \mathbf{m'_1}[\mathbf{m''}/\mathbf{x_1}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

Instantiating $(\mathbf{W_2}, \mathbf{m'_1}[\mathbf{m''}/\mathbf{x_1}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$ with $\mathbf{E}^{\not\uparrow}$, we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'_1}[\mathbf{m''}/\mathbf{x_1}]]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\mathbf{i} = 2$

Note that $(\mathbf{W_2}, \mathbf{m''}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta}$

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma}, \mathbf{x_2} : \mathbf{t}_2 \vdash \mathbf{m_2} : \mathbf{t}$, we know $\mathbf{\Sigma_D}; \mathbf{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_2} : \mathbf{t}_2 \vDash^{\Uparrow} \mathbf{m_2} : \mathbf{t}$.

We instantiate this with $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\mathbf{\Delta}]\!]^{\Sigma}$, $(\mathbf{W_2}, \boldsymbol{\gamma}\,[\mathbf{x_2} \mapsto \mathbf{m''}]) \in \mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}, \mathbf{x_2} : \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, $(\mathbf{W_2}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Therefore we know $(\mathbf{W_2}, \boldsymbol{\gamma}\,[\mathbf{x_2} \mapsto \mathbf{m''}]\,(\mathbf{m'_2})) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$. Note that this is exactly $(\mathbf{W_2}, \mathbf{m'_2}[\mathbf{m''}/\mathbf{x_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

Instantiating $(\mathbf{W_2}, \mathbf{m'_2}[\mathbf{m''}/\mathbf{x_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$ with $\mathbf{E}^{\not\uparrow}$, we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'_2}[\mathbf{m''}/\mathbf{x_2}]]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case $\exists \mathsf{s}_2^{\hat{+}}.\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$, but $\nexists \mathsf{s}_1^{\hat{+}}.\mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$.

The proof of this case is symmetric to the previous case. The primary difference is the cases for $\mathbf{i} = 1$ and $\mathbf{i} = 2$ are switched.

Case $\nexists \mathsf{s}_1^{\hat{+}}, \mathsf{s}_2^{\hat{+}}.\mathbf{t}_1 = \mathsf{s}_1^{\hat{+}}$ or $\mathbf{t}_2 = \mathsf{s}_2^{\hat{+}}$

This case is analogous to the previous cases. We repeat it since both types are non-translation types and it is not obviously symmetric.

We must show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[\mathbf{case\,m'\,of\,inj_1\;x_1.\,m'_1 \parallel inj_2\;x_2.\;m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_1 + \mathbf{t}_2$, we know $(\mathbf{W_1}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By $(\mathbf{W_1}, \mathbf{m'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[\mathbf{case}\,[\cdot]_{\mathbf{T}}\,\mathbf{of\,inj_1\;x_1.\,m'_1 \parallel inj_2\;x_2.\;m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{inj_i\;m''}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 + \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{case\,inj_i\;m''\,of\,inj_1\;x_1.\,m'_1 \parallel inj_2\;x_2.\;m'_2}]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By FD-Val, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m_i}[\mathbf{m''}/\mathbf{x_i}]]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x_i} : \mathbf{t}_i, \mathbf{\Gamma} \vdash \mathbf{m_i} : \mathbf{t}$, we know $(\mathbf{W_2}, \boldsymbol{\gamma}\,[\mathbf{x_i} \mapsto \mathbf{m''}]\,(\mathbf{m'_i})) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$. Note that this is exactly $(\mathbf{W_2}, \mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$.

Instantiating $(\mathbf{W_2}, \mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}]\!]^{\Sigma}_{\delta}$ with $\mathbf{E}^{\not\uparrow}$, we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m'_i}[\mathbf{m''}/\mathbf{x_i}]]) \in \mathcal{E}^{\Uparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.

Case FT-Fun

$$\frac{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathbf{t}_1, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_2 \qquad \mathbf{D}_\ell, \mathbf{\Delta} \vdash \mathbf{t}_1 :: *}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \lambda \mathbf{x}{:}\mathbf{t}_1.\mathbf{m} : \mathbf{t}_1 \to \mathbf{t}_2}$$

We must show $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vDash^{\hat{\uparrow}} \lambda \mathbf{x}{:}\mathbf{t}_1.\mathbf{m} : \mathbf{t}_1 \to \mathbf{t}_2$.

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \boldsymbol{\gamma}$. such that $\delta \in \mathcal{D}^{\hat{\uparrow}}[\![\mathbf{\Delta}]\!]^{\mathbf{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_\delta$, we must show
$(\mathbf{W}, \delta(\gamma(\boldsymbol{\gamma}(\lambda \mathbf{x}{:}\mathbf{t}_1.\mathbf{m})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.
For brevity, let $\mathbf{m}' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$, and $\mathbf{t}_1' = \delta(\mathbf{t}_1)$.

Case $\exists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$, and $\exists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$

We must show $(\mathbf{W}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathsf{s}_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathbf{t}_1, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_2$ we know: $((\mathbf{W}, \mathbf{x} : \mathsf{s}_1^{\hat{+}}), \mathbf{m}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By FD-Fun, we know $(\mathbf{W}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

Case $\exists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$, but $\nexists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$

We must show $(\mathbf{W}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.

For arbitrary $\mathbf{W_1} \sqsupseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not\downarrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\downarrow}[\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By the definition of $\mathcal{K}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathbf{t}_2, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show $(\mathbf{W_1}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.

By the definition of $\mathcal{V}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}} \to \mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show, for arbitrary $\mathbf{W_2} \sqsupseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m_1}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, that $(\mathbf{W_2}, \mathbf{m}'[\mathbf{x}/\mathbf{m_1}]) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$, which follows by the IH applied to
$\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}_1^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}_2$, with $(\mathbf{W_2}, \gamma\,[\mathbf{x} \mapsto \mathbf{m_1}]) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}, \mathbf{x} : \mathsf{s}_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

Case $\exists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$, but $\nexists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$

We must show $(\mathbf{W}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

For arbitrary $\mathbf{W_1} \sqsupseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not\downarrow}) \in \mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathsf{s}_2^{\hat{+}}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\downarrow}[\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}']) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By the definition of $\mathcal{K}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathsf{s}_2^{\hat{+}}, \mathsf{s}^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show $(\mathbf{W_1}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}') \in \mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$.

By the definition of $\mathcal{V}^{\hat{\uparrow}}[\![\mathbf{t}_1 \to \mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, it suffices to show, for arbitrary $\mathbf{W_2} \sqsupseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{m_1}) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1]\!]^{\mathbf{\Sigma}}_\delta$, that $(\mathbf{W_2}, \mathbf{m}'[\mathbf{x}/\mathbf{m_1}]) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathsf{s}_2^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, which follows by the IH applied to
$\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma}, \mathbf{x} : \mathsf{s}_1^{\hat{+}} \vdash \mathbf{m} : \mathbf{t}_2$, with $(\mathbf{W_2}, \boldsymbol{\gamma}\,[\mathbf{x} \mapsto \mathbf{m_1}]) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}, \mathbf{x} : \mathbf{t}_1]\!]^{\mathbf{\Sigma}}_\delta$.

Case $\nexists \mathsf{s}_1.\mathbf{t}_1 = \mathsf{s}_1$, and $\nexists \mathsf{s}_2.\mathbf{t}_2 = \mathsf{s}_2$

This case is exactly analogous to the previous cases.

Case FT-App

$$\frac{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \to \mathbf{t}_2 \qquad \mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_1}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1}\ \mathbf{m_2} : \mathbf{t}_2}$$

We must show $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vDash^{\hat{\uparrow}} \mathbf{m_1}\ \mathbf{m_2} : \mathbf{t}_2$.

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \boldsymbol{\gamma}$. such that $\delta \in \mathcal{D}^{\hat{\uparrow}}[\![\mathbf{\Delta}]\!]^{\mathbf{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma_1^{\hat{+}}]\!]^{\mathbf{\Sigma}}_\delta$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\uparrow}}[\![\mathbf{\Gamma}]\!]^{\mathbf{\Sigma}}_\delta$, we must show
$(\mathbf{W}, \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m_1}\ \mathbf{m_2})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_2]\!]^{\mathbf{\Sigma}}_\delta$.
For brevity, let $\mathbf{m_1'} = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m_1})))$, and $\mathbf{m_2'} = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m_2})))$.

Case $\exists s_2.t_2 = s_2^{\hat{+}}$

Case $\exists s_1.t_1 = s_1^{\hat{+}}$.

We must show
$(\mathbf{W}, \mathbf{m_1'}\ \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By FD-Subterm and FD-App, it suffices to show $(\mathbf{W}, \mathbf{m_1'}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}} \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$ and $(\mathbf{W}, \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![s_1^{\hat{+}}]\!]_{\delta}^{\Sigma}$, which follow by the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \to \mathbf{t}_2$ and $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_1$

Case $\nexists s_1.t_1 = s_1^{\hat{+}}$.

We must show
$(\mathbf{W}, \mathbf{m_1'}\ \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \to s_2^{\hat{+}}$, we know $(\mathbf{W}, \mathbf{m_1'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By definition of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W}, [\cdot]_{\mathbf{T}}\ \mathbf{m_2'}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

For arbitrary $\mathbf{W'} \sqsupseteq \mathbf{W}$, $(\mathbf{W'}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} \to \mathbf{t}_2]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W'}, (\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m})\mathbf{m_2'}) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

Note that $(\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m})\ \mathbf{m_2'} = \mathbf{E}^{\not\uparrow}[\mathbf{u}]$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_1$, we know $(\mathbf{W'}, \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1]\!]_{\delta}^{\Sigma}$.

By FD-Val and the definition of $\mathcal{E}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W'}, \mathbf{m}[\mathbf{m_2'}/\mathbf{x}]) \in \mathcal{E}^{\Uparrow}[\![s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$, which follows by instantiating $(\mathbf{W'}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$ with $(\mathbf{W'}, \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1]\!]_{\delta}^{\Sigma}$.

Case $\nexists s_2.t_2 = s_2^{\hat{+}}$

We must show
$(\mathbf{W}, \mathbf{m_1'}\ \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]_{\delta}^{\Sigma}$.

For arbitrary $\mathbf{W_1} \sqsupseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_2, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffice to show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[\mathbf{m_1'}\ \mathbf{m_2'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_1} : \mathbf{t}_1 \to s_2^{\hat{+}}$, we know $(\mathbf{W_1}, \mathbf{m_1'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By definition of $\mathcal{E}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}[[\cdot]_{\mathbf{T}}\ \mathbf{m_2'}]) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

For arbitrary $\mathbf{W_2} \sqsupseteq \mathbf{W_1}$, $(\mathbf{W_2}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![s_1^{\hat{+}} \to \mathbf{t}_2]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[(\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m})\mathbf{m_2'}]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

Note that $\mathbf{E}^{\not\uparrow}[(\lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m})\ \mathbf{m_2'}] = \mathbf{E_1}^{\not\uparrow}[\mathbf{u}]$.

By the IH applied to $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m_2} : \mathbf{t}_1$, we know $(\mathbf{W_2}, \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1]\!]_{\delta}^{\Sigma}$.

By FD-Val and the definition of $\mathcal{E}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}[\mathbf{m}[\mathbf{m_2'}/\mathbf{x}]]) \in \mathcal{O}^{\Uparrow}[\![s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By Lemma 5.27 applied to $(\mathbf{W_1}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_2, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$, we conclude $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_2, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$.

By $(\mathbf{W_2}, \mathbf{E}^{\not\uparrow}) \in \mathcal{K}^{\Uparrow}[\![\mathbf{t}_2, s^{\hat{+}}]\!]_{\delta}^{\Sigma}$, it suffices to show $(\mathbf{W_2}, \mathbf{m}[\mathbf{m_2'}/\mathbf{x}]) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_2]\!]_{\delta}^{\Sigma}$, which follows by instantiating $(\mathbf{W_2}, \lambda \mathbf{x}{:}\mathbf{t}_1'.\mathbf{m}) \in \mathcal{V}^{\Uparrow}[\![\mathbf{t}_1 \to s_2^{\hat{+}}]\!]_{\delta}^{\Sigma}$ with $(\mathbf{W_2}, \mathbf{m_2'}) \in \mathcal{E}^{\Uparrow}[\![\mathbf{t}_1]\!]_{\delta}^{\Sigma}$.

Case FT-Abs

$$\frac{\mathbf{D}_\ell, \boldsymbol{\alpha} :: \boldsymbol{\kappa}, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{m} : \mathbf{t}}{\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vdash \mathbf{\Lambda}\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\mathbf{m} : \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}}$$

We must show $\mathbf{D}_\ell, \mathbf{\Delta}; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{\Gamma} \vDash^{\Uparrow} \mathbf{\Lambda}\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\mathbf{m} : \forall \boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\, \mathbf{t}$

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \delta, \gamma, \boldsymbol{\gamma}$. such that $\boldsymbol{\delta} \in \mathcal{D}^{\Uparrow}[\![\mathbf{\Delta}]\!]^{\Sigma}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\Uparrow}[\![\Gamma_1^{\hat{+}}]\!]_{\delta}^{\Sigma}$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\Uparrow}[\![\mathbf{\Gamma}]\!]_{\delta}^{\Sigma}$, we must show

$(\mathbf{W}, \delta(\gamma(\boldsymbol{\gamma}(\boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m})))) \in \mathcal{E}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
For brevity, let $\mathbf{m}' = \delta(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$.

Case $\exists \mathsf{T}_\ell\ \mathsf{s}.\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t} = (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}$.

We must show $(\mathbf{W}, \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}) \in \mathcal{E}^{\twoheaduparrow}[\![(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
By FD-Return, it suffices to show $((\mathbf{W_k}, \mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}; \mathbf{W}_\Gamma), \mathbf{m}'[(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}/\boldsymbol{\alpha}]\langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{E}^{\twoheaduparrow}[\![(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$
Let $\mathbf{m}'[(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}/\boldsymbol{\alpha}] = \mathbf{m}''$.
Let $\delta' = \delta\left[\boldsymbol{\alpha} \mapsto ((\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}, \mathcal{E}^{\twoheaduparrow}[\![(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta})\right]$.
By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\alpha} :: *, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : ((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}$, we conclude $(\mathbf{W}, \mathbf{m}'') \in \mathcal{E}^{\twoheaduparrow}[\![((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$.
By Lemma 5.26 (Monotonicity of $\mathcal{E}^{\twoheaduparrow}[\![\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}$), we know
$((\mathbf{W_k}, \mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}; \mathbf{W}_\Gamma), \mathbf{m}'') \in \mathcal{E}^{\twoheaduparrow}[\![((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$.
Therefore, it suffices to show
$((\mathbf{W_k}, \mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}; \mathbf{W}_\Gamma), [\cdot]_{\mathbf{T}}\langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{K}^{\twoheaduparrow}[\![((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}, \delta']\!]^{\boldsymbol{\Sigma}}_{\delta}$.
For arbitrary $\mathbf{W_1} \supseteq (\mathbf{W_k}, \mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}; \mathbf{W}_\Gamma)$,
$(\mathbf{W_1}, \boldsymbol{\lambda}\mathbf{x}{:}(\ldots).\mathbf{m_1}) \in \mathcal{V}^{\twoheaduparrow}[\![((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$,
show $(\mathbf{W_1}, (\boldsymbol{\lambda}\mathbf{x}{:}(\ldots).\mathbf{m_1})\ \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{E}^{\twoheaduparrow}[\![\boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$.
By definition of $\mathcal{E}^{\twoheaduparrow}[\![\boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}$, it suffices to show
$(\mathbf{W_1}, (\boldsymbol{\lambda}\mathbf{x}{:}(\ldots).\mathbf{m_1})\ \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{E}^{\twoheaduparrow}[\![(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
By FD-Val, it suffices to show $(\mathbf{W_1}, \mathbf{m_1}[\langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle/\mathbf{x}]) \in \mathcal{E}^{\twoheaduparrow}[\![(\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
From $(\mathbf{W_1}, \boldsymbol{\lambda}\mathbf{x}{:}(\ldots).\mathbf{m}) \in \mathcal{V}^{\twoheaduparrow}[\![((\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})) \to \boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$ and definition of $\mathcal{E}^{\twoheaduparrow}[\![\boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}$, it suffices to show
$(\mathbf{W_1}, \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{E}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$.
For arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{W_2}, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha}), \delta']\!]^{\boldsymbol{\Sigma}}_{\delta}$, we must show $(\mathbf{W_2}, \mathbf{E}^{\not\dagger}[\langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle]) \in \mathcal{O}^{\twoheaduparrow}[\![\delta']\!]^{\boldsymbol{\Sigma}}_{\delta}$.
By assumption that $(\mathbf{W_2}, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha}), \delta']\!]^{\boldsymbol{\Sigma}}_{\delta}$, it suffices to show $(\mathbf{W_2}, \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{V}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$.
The proof that $(\mathbf{W_2}, \mathbf{k}) \in \mathcal{E}^{\twoheaduparrow}[\![\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha}]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$ is similar to the the FT-Var case for $\mathbf{x} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}$.
Recall that the type of $\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!]$ is $(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}})$, which cannot appear in evaluation position. Therefore, $(\mathbf{W_2}, \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!]) \in \mathcal{E}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha})]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$ is vacuously true.
Therefore, $(\mathbf{W_2}, \langle\hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!],\mathbf{k}\rangle) \in \mathcal{V}^{\twoheaduparrow}[\![(\hat{\boldsymbol{\alpha}}_\preceq\ \ \hat{\boldsymbol{\alpha}}_\ell\ \ \boldsymbol{\alpha}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\alpha})]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta'$, completing this case of the proof.

Case $\nexists \mathsf{T}_\ell\ \mathsf{s}.\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t} = \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}}$.

We must show $(\mathbf{W}, \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}') \in \mathcal{E}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
For arbitrary $\mathbf{W_1} \supseteq \mathbf{W}$, $(\mathbf{W_1}, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$, we must show $(\mathbf{W_1}, \mathbf{E}^{\not\dagger}[\boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}']) \in \mathcal{O}^{\twoheaduparrow}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
By definition of $(\mathbf{W_1}, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\delta}$, it suffices to show $(\mathbf{W_1}, \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}') \in \mathcal{V}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}$.
To show $(\mathbf{W_1}, \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}') \in \mathcal{V}^{\twoheaduparrow}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\delta}$, it suffices to show, for arbitrary $\mathbf{W_2} \supseteq \mathbf{W_1}$, $(\mathbf{t_1}, \mathbf{R}) \in \mathit{Rel}^{\twoheaduparrow}{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$, $(\mathbf{W_2}, \mathbf{m}'[\mathbf{t_1}/\boldsymbol{\alpha}]) \in \mathcal{E}^{\twoheaduparrow}[\![\mathbf{t}[\mathbf{t_1}/\boldsymbol{\alpha}]]\!]^{\boldsymbol{\Sigma}}_{\delta}\delta\,[\boldsymbol{\alpha} \mapsto (\mathbf{t}', \mathbf{R})]$, which follows by IH applied to $\mathbf{D}_\ell, \boldsymbol{\alpha} :: \boldsymbol{\kappa}, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G_k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m}' : \mathbf{t}$.

Case FT-Inst We know

$$\frac{\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}_1 \qquad \mathbf{D}_\ell, \boldsymbol{\Delta} \vdash \mathbf{t}_2 :: \boldsymbol{\kappa}}{\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m}\,[\mathbf{t}_2] : \mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}]}$$

For arbitrary $\Gamma_1^{\hat{+}}, \Gamma_2^{\hat{+}}, \boldsymbol{\delta}, \gamma, \boldsymbol{\gamma}$. such that $\boldsymbol{\delta} \in \mathcal{D}^{\hat{\dagger}}[\![\boldsymbol{\Delta}]\!]^{\boldsymbol{\Sigma}}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\dagger}}[\![\Gamma_1^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, $(\mathbf{W}, \boldsymbol{\gamma}) \in \mathcal{G}^{\hat{\dagger}}[\![\boldsymbol{\Gamma}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show
$(\mathbf{W}, \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{m}\,[\mathbf{t}_2])))) \in \mathcal{E}^{\hat{\dagger}}[\![\mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}]]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For brevity, let $\mathbf{m}' = \boldsymbol{\delta}(\gamma(\boldsymbol{\gamma}(\mathbf{m})))$.

Case $\exists \mathsf{s}.\mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}] = \mathsf{s}^{\hat{+}}$.

We must show $(\mathbf{W}, \mathbf{m}\,[\mathbf{t}_2]) \in \mathcal{E}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
Note then that $\nexists \mathsf{s}.\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t} = (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}$.
By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}$, we know $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}, [\cdot]_\mathbf{T}\,[\mathbf{t}_2]) \in \mathcal{K}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}$, $(\mathbf{W}', \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}'') \in \mathcal{V}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}', (\boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}'')\,[\mathbf{t}_2]) \in \mathcal{O}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
Note that $[\cdot]_\mathbf{T}\,[\mathbf{t}_2] = \mathbf{E}^{\not\dagger}$. Therefore, by FD-Val, it suffices to show $(\mathbf{W}', \mathbf{m}''[\mathbf{t}_2/\boldsymbol{\alpha}]) \in \mathcal{E}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By Lemma 5.31 (back-translation type interpretation is well-formed) applied to $\boldsymbol{\Delta} \vdash \mathbf{t}_2 :: \boldsymbol{\kappa}$ we know $(\mathbf{t}_2, \mathcal{T}^{\hat{\dagger}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}) \in Rel^{\hat{\dagger}}\,{}^{\boldsymbol{\Sigma}}_{\boldsymbol{\kappa}}$.
Instantiating $(\mathbf{W}', \boldsymbol{\Lambda\alpha}{::}\boldsymbol{\kappa}.\mathbf{m}'') \in \mathcal{V}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$ with $(\mathbf{t}_2, \mathcal{T}^{\hat{\dagger}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa}_1 \to \boldsymbol{\kappa}_2]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}})$, we conclude $(\mathbf{W}', \mathbf{m}''[\mathbf{t}_2/\boldsymbol{\alpha}]) \in \mathcal{E}^{\hat{\dagger}}[\![\mathbf{t}_1]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}\boldsymbol{\delta}\left[\boldsymbol{\alpha} \mapsto (\mathbf{t}_2, \mathcal{T}^{\hat{\dagger}}[\![\mathbf{t}_2 :: \boldsymbol{\kappa}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}})\right]$.
By Lemma 5.32 (Compositionality), we conclude $(\mathbf{W}', \mathbf{m}''[\mathbf{t}_2/\boldsymbol{\alpha}]) \in \mathcal{E}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\nexists \mathsf{s}.\mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}] = \mathsf{s}^{\hat{+}}$.

We must show $(\mathbf{W}, \mathbf{m}'\,[\mathbf{t}_2]) \in \mathcal{E}^{\hat{\dagger}}[\![\mathbf{t}_1[\mathbf{t}_2/\boldsymbol{\alpha}]]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}_1 \sqsupseteq \mathbf{W}$, $(\mathbf{W}_1, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\hat{\dagger}}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}_1, \mathbf{E}^{\not\dagger}[\mathbf{m}'\,[\mathbf{t}_2]]) \in \mathcal{O}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\exists \mathsf{s}.\forall\boldsymbol{\alpha}{::}{*}.\,\mathbf{t}_1 = (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}$.

Then $\mathbf{E}^{\not\dagger}$ must equal $[\cdot]_\mathbf{T}\mathbf{m}''$ where $\boldsymbol{\Sigma}_\mathbf{D}; \boldsymbol{\Sigma}_\mathbf{G}, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}} \vdash \mathbf{m}'' : (\hat{\alpha}_\preceq\ \hat{\alpha}_\ell\ (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell \mathsf{s})^{\hat{+}})$.
By FD-Bind, it suffices to show $(\mathbf{W}_1, \mathbf{m}') \in \mathcal{O}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follows by IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \forall\boldsymbol{\alpha}{::}{*}.\,\mathbf{t}_1$, and $(\mathbf{W}_1, \mathbf{prj}_2\ \mathbf{m}_\mathbf{p}) \in \mathcal{O}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
By $(\mathbf{W}_1, \mathbf{E}^{\not\dagger}) \in \mathcal{K}^{\hat{\dagger}}[\![\mathbf{t}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, it suffices to show $(\mathbf{W}_1, \boldsymbol{\lambda}\mathbf{x}{:}(\hat{\alpha}_\preceq\ \hat{\alpha}_\ell\ (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell \mathsf{s})^{\hat{+}}).\mathbf{prj}_2\mathbf{x}) \in \mathcal{V}^{\hat{\dagger}}[\![(\cdots)]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}_2 \sqsupseteq \mathbf{W}_1$, $(\mathbf{W}_2, \mathbf{m}_1) \in \mathcal{E}^{\hat{\dagger}}[\![(\hat{\alpha}_\preceq\ \hat{\alpha}_\ell\ (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}_2, \mathbf{prj}_2\ \mathbf{m}_1) \in \mathcal{E}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
It suffices to show $(\mathbf{W}_2, \mathbf{prj}_2\,[\cdot]_\mathbf{T}) \in \mathcal{K}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}, \mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.
For arbitrary $\mathbf{W}' \sqsupseteq \mathbf{W}_2$, $(\mathbf{W}', \mathbf{m}_2) \in \mathcal{V}^{\hat{\dagger}}[\![(\hat{\alpha}_\preceq\ \hat{\alpha}_\ell\ (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, we must show $(\mathbf{W}', \mathbf{prj}_2\ \mathbf{m}_2) \in \mathcal{O}^{\hat{\dagger}}[\![\mathsf{s}^{\hat{+}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$, which follows from FD-Val and assumption that $(\mathbf{W}', \mathbf{m}_2) \in \mathcal{V}^{\hat{\dagger}}[\![(\hat{\alpha}_\preceq\ \hat{\alpha}_\ell\ (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}) \times (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Case $\nexists \mathsf{s}.\forall\boldsymbol{\alpha}{::}{*}.\,\mathbf{t}_1 = (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}$.

Note that $\mathbf{E}^{\not\dagger}[[\cdot]_\mathbf{T}\,[\mathbf{t}_2]] = \mathbf{E}_1^{\not\dagger}$.
By the IH applied to $\mathbf{D}_\ell, \boldsymbol{\Delta}; \mathbf{G}_\ell, \mathbf{G}_\preceq, \mathbf{G}_\mathbf{k}, \Gamma^{\hat{+}}, \boldsymbol{\Gamma} \vdash \mathbf{m} : \forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}_1$, we know $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\dagger}}[\![\forall\boldsymbol{\alpha}{::}\boldsymbol{\kappa}.\,\mathbf{t}_1]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}$.

Therefore, it suffices to show $(\mathbf{W}, \mathbf{E}^{\not+}[[\cdot]_{\mathbf{T}}\,[\mathbf{t}_2]]) \in \mathcal{K}^{\hat{\uparrow}}[\![\forall \alpha{::}\kappa.\, \mathbf{t}_1, \mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
For arbitrary $\mathbf{W}' \supseteq \mathbf{W}$, $(\mathbf{W}', \Lambda\alpha{::}\kappa.\mathbf{m}'') \in \mathcal{V}^{\hat{\uparrow}}[\![\forall\alpha{::}\kappa.\, \mathbf{t}_2]\!]^{\Sigma}_{\delta}$, we must show $(\mathbf{W}', \mathbf{E}^{\not+}[(\Lambda\alpha{::}\kappa.\mathbf{m}'')\,[\mathbf{t}_2]]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$.
By FD-Val, it suffices to show $(\mathbf{W}', \mathbf{E}^{\not+}[\mathbf{m}''[\mathbf{t}_2/\alpha]]) \in \mathcal{O}^{\hat{\uparrow}}[\![\mathsf{s}^{\hat{+}}]\!]^{\Sigma}_{\delta}$, which follows by $(\mathbf{W}', \Lambda\alpha{::}\kappa.\mathbf{m}'') \in \mathcal{V}^{\hat{\uparrow}}[\![\forall\alpha{::}\kappa.\, \mathbf{t}_1]\!]^{\Sigma}_{\delta}$, Lemma 5.32 (back-translation compositionality), and Lemma 5.31 (back-translation type interpretation is well-formed) applied to $\Delta \vdash \mathbf{t}_2 :: \kappa$.

Case FT-Eqv

We know

$$\frac{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \Gamma \vdash \mathbf{m} : \mathbf{t}_1 \qquad \mathbf{t}_1 \equiv \mathbf{t}_2 \qquad \mathbf{D}_\ell, \Delta \vdash \mathbf{t}_2 :: *}{\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \Gamma \vdash \mathbf{m} : \mathbf{t}_2}$$

For arbitrary $\Gamma^{\hat{+}}_{\mathbf{1}}, \Gamma^{\hat{+}}_{\mathbf{2}}, \delta, \gamma, \gamma$. such that $\delta \in \mathcal{D}^{\hat{\uparrow}}[\![\Delta]\!]^{\Sigma}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma^{\hat{+}}_{\mathbf{1}}]\!]^{\Sigma}_{\delta}$, $(\mathbf{W}, \gamma) \in \mathcal{G}^{\hat{\uparrow}}[\![\Gamma]\!]^{\Sigma}_{\delta}$, we must show
$(\mathbf{W}, \delta(\gamma(\gamma(\mathbf{m})))) \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta}$.
For brevity, let $\mathbf{m}' = \delta(\gamma(\gamma(\mathbf{m})))$.

We must show $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

By IH applied to $\mathbf{D}_\ell, \Delta; \mathbf{G}_\ell, \mathbf{G}_{\preceq}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}}, \Gamma \vdash \mathbf{m} : \mathbf{t}_1$, we know $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_1]\!]^{\Sigma}_{\delta}$.

By Theorem 5.33 (Coherence for back-translation relation), we conclude $(\mathbf{W}, \mathbf{m}') \in \mathcal{E}^{\hat{\uparrow}}[\![\mathbf{t}_2]\!]^{\Sigma}_{\delta}$.

□

Theorem 5.35 (Back-translation exists)
If $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}}$ *then* $\exists \mathsf{e}.\ \Sigma; \mathbf{G}_{\mathbf{k}}; \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$.

Proof

Follows from Lemma 5.34 □

Lemma 5.36 (Back-translation preserves semantics (generalized))
Let $\delta = \{\alpha_\ell \mapsto \hat{\alpha}_\ell \ \mid\ \ell \in \mathcal{L}_\ell\} \cup \{\alpha_{\preceq} \mapsto \hat{\alpha}_{\preceq}\}$, $\gamma_{\mathbf{k}} = \{\mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}} \mapsto \eta^{\ell,\mathsf{s}}_{\mathbf{k}} \mid \mathbf{k} : \mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}} \in \mathbf{G}_{\mathbf{k}}\}$.
If $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}}$ *and* $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$ *then* $\Gamma \mid \Sigma \vdash \mathsf{e} \simeq \gamma_{\mathbf{k}}(\mathbf{m}) : \mathsf{s} \mid \delta$

Proof

By induction on the structure of $\Sigma_{\mathbf{D}}; \Sigma_{\mathbf{G}}, \mathbf{G}_{\mathbf{k}}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$.

Case (FD-Unit) $\Sigma; \mathbf{G}_{\mathbf{k}}; \Gamma^{\hat{+}} \vdash \langle\rangle : \mathbf{1} \Uparrow \langle\rangle$

For arbitrary ζ, $(\gamma, \gamma) \in \mathcal{G}^{+}_{\zeta}[\![\Gamma]\!]_{\delta}$, must show $(\langle\rangle, \langle\rangle) \in \mathcal{E}^{+}_{\zeta}[\![1]\!]_{\delta}$.

Since $\langle\rangle$ and $\langle\rangle$ are values, it suffices to show $(\langle\rangle, \langle\rangle) \in \mathcal{V}^{+}_{\zeta}[\![1]\!]_{\delta}$, which follows by definition.

Case (FD-Var) $$\frac{(\mathbf{x} : \mathsf{s}^{\hat{+}}) \in \Gamma^{\hat{+}}}{\Sigma; \mathbf{G}_{\mathbf{k}}; \Gamma^{\hat{+}} \vdash \mathbf{x} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{x}}$$

For arbitrary ζ, $(\gamma, \gamma) \in \mathcal{G}^{+}_{\zeta}[\![\Gamma]\!]_{\delta}$, must show $(\gamma(\mathsf{x}), \gamma(\mathbf{x})) \in \mathcal{E}^{+}_{\zeta}[\![\mathsf{s}]\!]_{\delta}$, which follows immediately from $(\gamma, \gamma) \in \mathcal{G}^{+}_{\zeta}[\![\Gamma]\!]_{\delta}$.

Case (FD-K) $$\frac{\mathbf{k} : (\mathsf{s}^{\hat{+}} \to (\mathsf{T}_\ell\ \mathsf{s})^{\hat{+}}) \in \mathbf{G}_{\mathbf{k}}}{\Sigma; \mathbf{G}_{\mathbf{k}}; \Gamma^{\hat{+}} \vdash \mathbf{k} : \mathsf{s}^{\hat{+}} \to \mathsf{T}_\ell\ \mathsf{s}^{\hat{+}} \Uparrow \lambda \mathsf{x} : \mathsf{s}.\, \eta_\ell\, \mathsf{x}}$$

For arbitrary ζ, $(\gamma, \gamma) \in \mathcal{G}^{+}_{\zeta}[\![\Gamma]\!]_{\delta}$, must show $(\lambda \mathsf{x} : \mathsf{s}.\, \eta_\ell\, \mathsf{x}, \eta^{\ell,\mathsf{s}}_{\mathbf{k}}) \in \mathcal{E}^{+}_{\zeta}[\![\mathsf{s} \to \mathsf{T}_\ell\ \mathsf{s}]\!]_{\delta}$.

It suffices to show $(\lambda \mathsf{x} : \mathsf{s}.\, \eta_\ell\, \mathsf{x}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}) \in \mathcal{V}_\zeta^{+}[\![\mathsf{s} \to \mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$.

For arbitrary $(\mathsf{e}', \mathbf{m}') \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$, we must show $(\eta_\ell\, \mathsf{e}', \boldsymbol{\Lambda\beta}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\beta})).((\mathbf{prj_2 x})\ \mathbf{m}')) \in \mathcal{E}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$.

It suffices to show $(\eta_\ell\, \mathsf{e}', \boldsymbol{\Lambda\beta}{::}{*}.\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \boldsymbol{\beta}) \times (\mathsf{s}^{\hat{+}} \to \boldsymbol{\beta})).((\mathbf{prj_2\ x})\ \ \mathbf{m}')) \in \mathcal{V}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Let $\boldsymbol{\rho} = [\![\mathcal{L}_\ell^{+}]\!]_\zeta^{\boldsymbol{\Sigma}}$.

By $\mathcal{V}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$, it suffices to show $\exists \mathbf{m}''.\boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{m}'' : \mathsf{s}^{\hat{+}})$ such that

- $((\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ (^{\hat{+}}\mathsf{T}_\ell\, \mathsf{s}))\ \times\ \mathsf{s}^{\hat{+}}\ \to\ (^{\hat{+}}\mathsf{T}_\ell\, \mathsf{s})).((\mathbf{prj_2\ x})\ \mathbf{m}'))\ \langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}'')\ \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\, \mathsf{s})^{+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$
  By evaluation we see that $((\boldsymbol{\lambda}\mathbf{x}{:}((\boldsymbol{\alpha}_{\preceq}\ \boldsymbol{\alpha}_\ell\ (^{\hat{+}}\mathsf{T}_\ell \mathsf{s})) \times \mathsf{s}^{\hat{+}} \to (^{\hat{+}}\mathsf{T}_\ell \mathsf{s})).((\mathbf{prj_2 x})\mathbf{m}'))\langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m}'')$ is equivalent to $(\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}', \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}'')$.
  So, let $\mathbf{m}'' = \mathbf{m}'$. By Theorem 5.14, we know $(\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}', \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m}') \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\, \mathsf{s})^{+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.
- $(\mathsf{e}', \mathbf{m}'') \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}^{+}]\!]_{\boldsymbol{\delta}}$
  Since $\mathbf{m}'' = \mathbf{m}'$, this hold by assumption.

Case $\text{(FD-Sum)}\ \dfrac{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_i^{\hat{+}} \Uparrow \mathsf{e}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{inj_i\ m} : \mathsf{s}_1^{\hat{+}}\ +\ \mathsf{s}_2^{\hat{+}} \Uparrow \mathsf{inj_i\, e}}$

For arbitrary ζ, $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^{+}[\![\Gamma]\!]_{\boldsymbol{\delta}}$, we must show $(\mathsf{inj}_i\, \gamma(\mathsf{e}), \mathbf{inj}_i\ \boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m}))) \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}_1 + \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

Since $\mathsf{inj}_i\, \gamma(\mathsf{e})$ and $\mathbf{inj}_i\ \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))$ are values, it suffices to show $(\mathsf{inj}_i\, \gamma(\mathsf{e}), \mathbf{inj}_i\ \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))) \in \mathcal{V}_\zeta^{+}[\![\mathsf{s}_1 + \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

By definition of $\mathcal{V}_\zeta^{+}[\![\mathsf{s}_1 + \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$, it suffices to show $(\gamma(\mathsf{e}), \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))) \in \mathcal{E}_\zeta^{+}[\![\mathsf{s_i}]\!]_{\boldsymbol{\delta}}$, which follows by IH applied to $\boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_i^{\hat{+}}$.

Case $\text{(FD-Pair)}\ \dfrac{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_1} : \mathsf{s}_1^{\hat{+}} \Uparrow \mathsf{e}_1 \qquad \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \mathbf{m_2} : \mathsf{s}_2^{\hat{+}} \Uparrow \mathsf{e}_2}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \langle \mathbf{m_1}, \mathbf{m_2} \rangle : \mathsf{s}_1^{\hat{+}}\ \times\ \mathsf{s}_2^{\hat{+}} \Uparrow \langle \mathsf{e}_1, \mathsf{e}_2 \rangle}$

This case is analogous to the previous case.

Case $\text{(FD-Fun)}\ \dfrac{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_2^{\hat{+}} \Uparrow \mathsf{e}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}_1^{\hat{+}}.\mathbf{m} : \mathsf{s}_1^{\hat{+}}\ \to\ \mathsf{s}_2^{\hat{+}} \Uparrow \lambda \mathsf{x} : \mathsf{s}_1.\, \mathsf{e}}$

For arbitrary ζ, $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^{+}[\![\Gamma]\!]_{\boldsymbol{\delta}}$, we must show $(\lambda \mathsf{x} : \mathsf{s}_1.\, \gamma(\mathsf{e}), \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}_1^{\hat{+}}.\boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))) \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}_1 \to \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

Let $\mathsf{e}_2 = \gamma(\mathsf{e})$ and $\mathbf{m_2} = \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))$.

It suffices to show $(\lambda \mathsf{x} : \mathsf{s}_1.\, \mathsf{e}_2, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s}_1^{\hat{+}}.\mathbf{m_2}) \in \mathcal{V}_\zeta^{+}[\![\mathsf{s}_1 \to \mathsf{s}_2]\!]_{\boldsymbol{\delta}}$.

For arbitrary $(\mathsf{e}_1, \mathbf{m_1}) \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}_1]\!]_{\boldsymbol{\delta}}$, we must show $(\mathsf{e}_2[\mathsf{e}_1/\mathsf{x}], \mathbf{m_2}[\mathbf{m_1}/\mathbf{x}]) \in \mathcal{E}_\zeta^{+}[\![\mathsf{s}_2]\!]_{\boldsymbol{\delta}}$, which follows by IH applied to $\boldsymbol{\Sigma}_{\mathbf{D}}; \boldsymbol{\Sigma}_{\mathbf{G}}, \mathbf{G_k}, \Gamma^{\hat{+}}, \mathbf{x} : \mathsf{s}_1^{\hat{+}} \vdash \mathbf{m} : \mathsf{s}_1^{\hat{+}}\ \to\ \mathsf{s}_2^{\hat{+}}$.

Case $\text{(FD-Return)}\ \dfrac{\boldsymbol{\Sigma}; \mathbf{G_k}, \mathbf{k} : (\mathsf{s}^{\hat{+}}\ \to\ (\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}); \Gamma^{\hat{+}} \vdash \mathbf{m}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \mathbf{k}\rangle : \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m} : \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}}$

For arbitrary ζ, $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^{+}[\![\Gamma]\!]_{\boldsymbol{\delta}}$, we must show $(\gamma(\mathsf{e}), \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m})))) \in \mathcal{E}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}^{+}]\!]_{\boldsymbol{\delta}}$.

Let $\mathsf{e}' = \gamma(\mathsf{e})$ and $\mathbf{m}' = \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m})))$.

It suffices to show $(\mathsf{e}', ()\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m}') \in \mathcal{E}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}^{+}]\!]_{\boldsymbol{\delta}}$.

By IH applied to $\boldsymbol{\Sigma}; \mathbf{G_k}, \mathbf{k} : (\mathsf{s}^{\hat{+}}\ \to\ (\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}); \Gamma^{\hat{+}} \vdash \mathbf{m}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \mathbf{k}\rangle : \mathsf{T}_\ell\, \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e}$ we know $(\mathsf{e}', \mathbf{m_1}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle) \in \mathcal{E}_\zeta^{+}[\![\mathsf{T}_\ell\, \mathsf{s}]\!]_{\boldsymbol{\delta}}$.

By Lemma 5.22 (Cross Lang. Relation Respects Target Relation), it suffices to show $(\mathbf{m_1}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\, \mathsf{s}^{+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

By Lemma 5.18 (Free Theorem: $\eta_{\mathbf{k}}^{\ell,\mathsf{s}}$ shuffling), we know $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \mathbf{m_1}[(\mathsf{T}_\ell\, \mathsf{s})^{\hat{+}}/\boldsymbol{\beta}]\langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\, \mathsf{s}^{+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

By Lemma 5.7 (Target Relation Symmetric under Lattice Interp.), we know $(\mathbf{m_1}[(\mathsf{T}_\ell\ \mathsf{s})^{\hat{\dagger}}/\beta]\ \langle \hat{\mathbf{pf}}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle, \boldsymbol{\Lambda\beta{::}{*}.\mathbf{m_1}}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{T}_\ell\ \mathsf{s}^{+}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

Case

$$(\text{FD-Bind})\ \frac{\begin{array}{c}\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m_p} : (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \mathsf{s}^{\hat{\dagger}}) \times (\mathsf{s'}^{\hat{\dagger}} \rightarrow \mathsf{s}^{\hat{\dagger}}) \\ \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m} : (\mathsf{T}_\ell\ \mathsf{s'})^{\hat{\dagger}} \Uparrow \mathsf{e} \qquad \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{prj_2\ m_p} : \mathsf{s'}^{\hat{\dagger}} \rightarrow \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e'}\end{array}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m}\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m_p} : \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{bind}\ \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e'}\ \mathsf{x}}$$

For arbitrary ζ, $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}^{\dagger}_{\zeta}[\![\Gamma]\!]_{\boldsymbol{\delta}}$, we must show $(\gamma(\mathsf{bind x{=}e in e'x}), \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m}\ [\mathsf{s}^{\hat{\dagger}}]\mathbf{m_{pf}})))) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Note that since $\boldsymbol{\Sigma_D}; \boldsymbol{\Sigma_G}, \mathbf{G_k}, \Gamma^{\hat{\dagger}} \vdash \mathbf{prj_1\ m_{pf}} : (\boldsymbol{\alpha}_{\preceq}\ \ \boldsymbol{\alpha}_\ell\ \ \mathsf{s}^{\hat{\dagger}})$, by Lemma 5.24 we know $\ell \preceq \mathsf{s}$.

Let $\mathsf{bind}\ \mathsf{x} = \mathsf{e_1}\ \mathsf{in}\ \mathsf{e_2}\ \mathsf{x} = \gamma(\mathsf{bind}\ \mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e'}\ \mathsf{x})$ and $\mathbf{m_1}\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m'_{pf}} = \boldsymbol{\delta}(\boldsymbol{\gamma}(\boldsymbol{\gamma}_{\mathbf{k}}(\mathbf{m}\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m_{pf}})))$

By IH applied to $\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m} : (\mathsf{T}_\ell\ \mathsf{s'})^{\hat{\dagger}} \Uparrow \mathsf{e}$, we know $\mathsf{e_1} \longmapsto^* \eta_\ell\ \mathsf{e'_1}$, and $\mathbf{m_1} \longmapsto^* \boldsymbol{\Lambda\beta{::}{*}.\mathbf{m'_1}}$, and $(\eta_\ell\ \mathsf{e'_1}, \boldsymbol{\Lambda\beta{::}{*}.\mathbf{m'_1}}) \in \mathcal{V}^{\dagger}_{\zeta}[\![\mathsf{T}_\ell\ \mathsf{s'}]\!]_{\boldsymbol{\delta}}$.

Therefore, $\exists \mathbf{m''_1}.(\mathbf{m'_1}[({}^{\hat{\dagger}}\mathsf{T}_\ell\ \mathsf{s'})/\beta]\langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s'}]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m''_1}) \in \boldsymbol{\mathcal{E}}[\![({}^{+}\mathsf{T}_\ell\ \mathsf{s'})]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$ and $(\mathsf{e'_1}, \mathbf{m''_1}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s'}]\!]_{\boldsymbol{\delta}}$

By Lemma 5.22 (Cross Lang. Relation Respects Target Relation) and Lemma 5.18 (Free Theorem: $\eta_{\mathbf{k}}^{\ell,\mathsf{s}}$ shuffling), we know $(\eta_\ell\ \mathsf{e'_1}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m''_1}) \in \mathcal{V}^{\dagger}_{\zeta}[\![\mathsf{T}_\ell\ \mathsf{s'}]\!]_{\boldsymbol{\delta}}$.

By Lemma 5.18 (Free Theorem: $\eta_{\mathbf{k}}^{\ell,\mathsf{s}}$ shuffling) and Lemma 5.13 (Target Relation Transitivity) we know

$((\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m''_1})\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m'_{pf}}, \mathbf{m_1}\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m'_{pf}}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^{\hat{\dagger}}]\!]^{\boldsymbol{\Sigma}}_{\boldsymbol{\rho}}$.

Note that $\mathsf{bind}\ \mathsf{x} = \mathsf{e_1}\ \mathsf{in}\ \mathsf{e_2}\ \mathsf{x} \longmapsto^* \mathsf{e_2}\ \mathsf{e'_1}$, and $(\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m''_1})\ [\mathsf{s}^{\hat{\dagger}}]\ \mathbf{m'_{pf}} \longmapsto^* (\mathbf{prj_2\ m'_{pf}})\ \mathbf{m''_1}$.

Therefore, to show $(\mathsf{bind x{=}e_1 in e_2 x}, \mathbf{m_1}\ [\mathsf{s}^{\hat{\dagger}}]\mathbf{m'_{pf}}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$, it suffices to show $(\mathsf{e_2 e'_1}, \mathbf{prj_2 m'_{pf} m''_1}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Since $(\mathsf{e'_1}, \mathbf{m''_1}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s'}]\!]_{\boldsymbol{\delta}}$, it suffices to show $(\mathsf{e_2}, \mathbf{prj_2\ m'_{pf}}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s'} \rightarrow \mathsf{s}]\!]_{\boldsymbol{\delta}}$, which follows by IH applied to $\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{prj_2\ m_{pf}} : \mathsf{s'}^{\hat{\dagger}} \rightarrow \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e'}$.

Case

$$(\text{FD-Subterm})\ \frac{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{F} : \mathsf{s}_1^{\hat{\dagger}} \Rightarrow \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{F'} \qquad \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m} : \mathsf{s}_1^{\hat{\dagger}} \Uparrow \mathsf{e}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{F}[\mathbf{m}] : \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{F'}[\mathsf{e}]}$$

By inlining the possible cases for $\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{F} : \mathsf{s}_1^{\hat{\dagger}} \Rightarrow \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{F'}$, this case requires 3 subcases. Each follows simply by the IH.

Case

$$(\text{FD-Val})\ \frac{\mathbf{E}^{\not\dagger}[\mathbf{u}] \longmapsto \mathbf{m_1} \qquad \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m_1} : \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{e}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{E}^{\not\dagger}[\mathbf{u}] : \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{e}}$$

Follows by IH applied to $\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{m_1} : \mathsf{s}_2^{\hat{\dagger}} \Uparrow \mathsf{e}$ and evaluation.

Case

$$(\text{FD-Stuck})\ \frac{\begin{array}{c}\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{case\,m\,of\,inj_1\ x_1.\,m_1 \parallel inj_2\ x_2.\ \ m_2} : \mathbf{t} \text{ where } \nexists \mathsf{s'}.\mathbf{t} = \mathsf{s'}^{\hat{\dagger}} \\ \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{case\,m\,of\,inj_1\ y_1.\,E^{\not\dagger}[m_1[y_1/x_1]] \parallel inj_2\ y_2.\ \ E^{\not\dagger}[m_2[y_2/x_2]]} : \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e} \\ (\textit{fresh}\ \mathbf{y'_1}, \mathbf{y'_2})\end{array}}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{E}^{\not\dagger}[\mathbf{case\,m\,of\,inj_1\ x_1.\,m_1 \parallel inj_2\ x_2.\ \ m_2}] : \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e}}$$

Follows simply by IH applied to
$\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{\dagger}} \vdash \mathbf{case\,m\,of\,inj_1\ y_1.\,E^{\not\dagger}[m_1[y_1/x_1]] \parallel inj_2\ y_2.\ \ E^{\not\dagger}[m_2[y_2/x_2]]} : \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e}$ and evaluation.

$\square$

Theorem 5.37 (Back-translation preserves semantics)
Let $\boldsymbol{\delta} = \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \mid \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_{\preceq} \mapsto \hat{\boldsymbol{\alpha}}_{\preceq}\}$.
If $\boldsymbol{\Sigma} \vdash \mathbf{m} : \mathsf{s}^{\hat{\dagger}}$, *then* $\boldsymbol{\Sigma}; \vdash; \mathbf{m} : \mathsf{s}^{\hat{\dagger}} \Uparrow \mathsf{e}$ *and* $\forall \zeta.(\mathsf{e}, \mathbf{m}) \in \mathcal{E}^{\dagger}_{\zeta}[\![\mathsf{s}]\!]_{\boldsymbol{\delta}}$.

Proof

Follows from Lemma 5.36 and Theorem 5.35. $\square$

5.6 Equivalence Preservation

Lemma 5.38 (Preservation and Reflection of Closed Terms)
Let $\boldsymbol{\rho} = [\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$, $(\mathsf{e_1}, \mathbf{m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\rho_1}}$, *and* $(\mathsf{e_2}, \mathbf{m_2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\rho_2}}$. *Then:*

1a *If* $(\mathsf{e_1}, \mathsf{e_2}) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta$, *then* $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

1b *If* $(\mathbf{m_1}, \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$, *then* $(\mathsf{e_1}, \mathsf{e_2}) \in \mathcal{E}[\![\mathsf{s}]\!]_\zeta$.

2a *If* $(\mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\mathsf{s}]\!]_\zeta$, *then* $(\mathbf{u_1}, \mathbf{u_2}) \in \boldsymbol{\mathcal{V}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

2b *If* $(\mathbf{u_1}, \mathbf{u_2}) \in \boldsymbol{\mathcal{V}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$, *then* $(\mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\mathsf{s}]\!]_\zeta$.

Proof

Part *1* follows easily from part *2*.

Proof by induction on the structure of the type s.. There are two interesting cases.

Case *2a* $\mathsf{s_1} \to \mathsf{s_2}$

(1) Have: $(\lambda \mathsf{x} : \mathsf{s_1}.\, \mathsf{e_2}, \lambda \mathsf{x} : \mathsf{s_1}.\, \mathsf{e_2'}) \in \mathcal{V}[\![\mathsf{s_1} \to \mathsf{s_2}]\!]_\zeta$.

(2) Have: $(\lambda \mathsf{x} : \mathsf{s_1}.\, \mathsf{e_2}, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1} \to \mathsf{s_2}]\!]_{\boldsymbol{\rho_1}}$.

(3) Have: $(\lambda \mathsf{x} : \mathsf{s_1}.\, \mathsf{e_2'}, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2'}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1} \to \mathsf{s_2}]\!]_{\boldsymbol{\rho_2}}$.
Must show: $(\boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2}, \boldsymbol{\lambda}\mathbf{x}{:}\mathsf{s_1^+}.\mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1^+} \boldsymbol{\to} \mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$

(4) Consider arbitrary $\mathbf{m_1}$, $\mathbf{m_1'}$ such that $(\mathbf{m_1}, \mathbf{m_1'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.
Must show: $(\mathbf{m_2}[\mathbf{m_1}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$

(5) Have: $\exists \mathsf{e_1}.(\mathsf{e_1}, \mathbf{m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho_1}}$ from Theorem 5.37. (Back-translation preserves semantics)

(6) Have: $\exists \mathsf{e_1'}.(\mathsf{e_1'}, \mathbf{m_1'}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho_2}}$ from Theorem 5.37 (Back-translation preserves semantics).

(7) Have: $(\mathsf{e_1}, \mathsf{e_1'}) \in \mathcal{E}[\![\mathsf{s_1}]\!]_\zeta$ by IH (*1b*) apply to $\mathsf{s_1}$.
Instantiating 1 with 7: $(\mathsf{e_2}[\mathsf{e_1}/\mathsf{x}], \mathsf{e_2'}[\mathsf{e_1'}/\mathsf{x}]) \in \mathcal{E}[\![\mathsf{s_2}]\!]_\zeta$
Instantiating 2 with 5: $(\mathsf{e_2}[\mathsf{e_1}/\mathsf{x}], \mathbf{m_2}[\mathbf{m_1}/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\![\mathsf{s_2}]\!]_{\boldsymbol{\rho_1}}$
Instantiating 3 with 6: $(\mathsf{e_2'}[\mathsf{e_1'}/\mathsf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \mathcal{E}_\zeta^+[\![\mathsf{s_2}]\!]_{\boldsymbol{\rho_2}}$
Proceed by induction (*1a*): $(\mathbf{m_2}[\mathbf{m_1}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m_1'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_2^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$

Case *2a* $\mathsf{T}_\ell\, \mathsf{s_1}$

(1) Have: $(\eta_\ell\, \mathsf{e_2}, \eta_\ell\, \mathsf{e_2'}) \in \mathcal{E}[\![\mathsf{T}_\ell\, \mathsf{s_1}]\!]_\zeta$.

(2) Have: $(\eta_\ell\, \mathsf{e_2}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s_1}]\!]_{\boldsymbol{\rho_1}}$.

(3) Have: $(\eta_\ell\, \mathsf{e_2'}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1'}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\, \mathsf{s_1}]\!]_{\boldsymbol{\rho_2}}$.
Must show: $(\boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1}, \boldsymbol{\Lambda\beta}{::}{*}.\mathbf{m_1'}) \in \boldsymbol{\mathcal{V}}[\![(\mathsf{T}_\ell\, \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$
To show this, there are two cases.

$\ell \sqsubseteq \zeta$ This case is indirect, *i.e.*, doesn't use the induction hypothesis.
Have: $(\mathsf{e_2}, \mathsf{e_2'}) \in \mathcal{E}[\![\mathsf{s_1}]\!]_\zeta$ from 1.
Have: $\exists \mathbf{m_2}.(\mathbf{m_1}[(\mathsf{T}_\ell\, \mathsf{s_1})^+/\boldsymbol{\beta}]\langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \boldsymbol{\eta}_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \boldsymbol{\eta}_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\, \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e_2}, \mathbf{m_2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho_1}}$ from 2.
Have: $\exists \mathbf{m_2'}.(\mathbf{m_1'}[(\mathsf{T}_\ell\, \mathsf{s_1})^+/\boldsymbol{\beta}]\langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\, \mathsf{s}]\!], \boldsymbol{\eta}_{\mathbf{k}}^{\ell,\mathsf{s}}\rangle, \boldsymbol{\eta}_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\, \mathsf{s_1})^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e_2'}, \mathbf{m_2'}) \in \mathcal{E}_\zeta^+[\![\mathsf{s_1}]\!]_{\boldsymbol{\rho_2}}$ from 3.
Therefore: $(\mathbf{m_2}, \mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s_1^+}]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$.

Have: $(\mathbf{\Lambda\beta{::}*.m_1}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ by Transitivity (Lemma 5.13) and $\eta_{\mathbf{k}}^{\ell,\mathsf{s}}$ shuffling (Lemma 5.18).

Have: $(\mathbf{\Lambda\beta{::}*.m_1'}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ by Transitivity (Lemma 5.13) and $\eta_{\mathbf{k}}^{\ell,\mathsf{s}}$ shuffling (Lemma 5.18).

Have: $(\eta_{\mathbf{k}}^{\ell,\mathsf{s}}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}_1^+ \to (\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ by Fundamental Property (Theorem 5.14).

Therefore: $(\eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$.

Have: $(\mathbf{\Lambda\beta{::}*.m_1}, \mathbf{\Lambda\beta{::}*.m_1'} \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma})$ by Transitivity (Lemma 5.13) and Symmetry (Lemma 5.7).

$\ell \not\sqsubseteq \zeta$ Consider arbitrary $\boldsymbol\pi$ such that $\boldsymbol\pi \in Rel_{\boldsymbol *}^{\boldsymbol\Sigma}$.

Must show: $(\mathbf{m_1}[\boldsymbol{\pi_1}/\boldsymbol\beta], \mathbf{m_1'}[\boldsymbol{\pi_2}/\boldsymbol\beta]) \in \boldsymbol{\mathcal{E}}[\![((\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol\beta) \times (\mathsf{s}_1^+ \to \boldsymbol\beta)) \to \boldsymbol\beta]\!]_{\boldsymbol\rho[\boldsymbol\alpha::\boldsymbol\kappa\mapsto\boldsymbol\pi]}^{\boldsymbol\Sigma}$

Have: $\mathbf{m_1}[\boldsymbol{\pi_1}/\boldsymbol\beta] \longmapsto^* \mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol{\pi_1}) \times (\mathsf{s}_1^+ \to \boldsymbol{\pi_1}).\mathbf{m_2}$.

Have: $\mathbf{m_1'}[\boldsymbol{\pi_2}/\boldsymbol\beta] \longmapsto^* \mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol{\pi_2}) \times (\mathsf{s}_1^+ \to \boldsymbol{\pi_2}).\mathbf{m_2'}$.

Let $\boldsymbol\rho' = \boldsymbol\rho[\boldsymbol\alpha :: \boldsymbol\kappa \mapsto \boldsymbol\pi]$.

Consider arbitrary $\mathbf{m}, \mathbf{m'}$ such that $(\mathbf{m}, \mathbf{m'}) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol\beta) \times (\mathsf{s}_1^+ \to \boldsymbol\beta)]\!]_{\boldsymbol\rho'}^{\boldsymbol\Sigma}$.

Must show: $(\mathbf{m_2}[\mathbf{m}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\boldsymbol\beta]\!]_{\boldsymbol\rho'}^{\boldsymbol\Sigma}$

By definition of $\boldsymbol\rho$, namely the relation on $\boldsymbol{\alpha_\preceq}$, we know $\boldsymbol{\pi_{\mathbf{R}}} = Atom\,[\boldsymbol{\pi_1}, \boldsymbol{\pi_2}]^{\boldsymbol\Sigma}$

Therefore: $(\mathbf{m_2}[\mathbf{m}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m'}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\boldsymbol\beta]\!]_{\boldsymbol\rho'}^{\boldsymbol\Sigma}$.

Case *2b* $\mathsf{T}_\ell\ \mathsf{s}_1$

(1) Have: $(\mathbf{\Lambda\beta{::}*.m_1}, \mathbf{\Lambda\beta{::}*.m_1'}) \in \boldsymbol{\mathcal{V}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$.

(2) Have: $(\eta_\ell\ \mathsf{e}_2, \mathbf{\Lambda\beta{::}*.m_1}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\ \mathsf{s}_1]\!]_{\boldsymbol{\rho_1}}$.

(3) Have: $(\eta_\ell\ \mathsf{e}_2', \mathbf{\Lambda\beta{::}*.m_1'}) \in \mathcal{E}_\zeta^+[\![\mathsf{T}_\ell\ \mathsf{s}_1]\!]_{\boldsymbol{\rho_1}}$.

Must show: $(\eta_\ell\ \mathsf{e}_2, \eta_\ell\ \mathsf{e}_2') \in \mathcal{V}[\![\mathsf{T}_\ell\ \mathsf{s}_1]\!]_\zeta$

Assume that $\ell \sqsubseteq \zeta$:

Must show: $(\mathsf{e}_2, \mathsf{e}_2') \in \mathcal{E}[\![\mathsf{s}_1]\!]_\zeta$

Let $\mathbf{R} = \{(\mathbf{m_1}, \mathbf{m_2}) \in Atom\,[(\mathsf{T}_\ell\ \mathsf{s}_1)^+]_{\boldsymbol\rho}^{\boldsymbol\Sigma} \mid$
$\ell \sqsubseteq \zeta \implies (\forall \mathbf{m'}, \mathbf{m''}.((\mathbf{m_1}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma} \wedge$
$(\mathbf{m_2}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \mathbf{m''}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}) \implies$
$(\mathbf{m'}, \mathbf{m''}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}_1^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma})\}$.

Let $\boldsymbol\pi = (\boldsymbol{\rho_1}((\mathsf{T}_\ell\ \mathsf{s}_1)^+), \boldsymbol{\rho_2}((\mathsf{T}_\ell\ \mathsf{s}_1)^+), \mathbf{R})$.

Instantiating 1 with $\boldsymbol\pi$:

Have: $\mathbf{m_1}[\boldsymbol{\rho_1}((\mathsf{T}_\ell\ \mathsf{s}_1)^+)/\boldsymbol\beta] \longmapsto^* \mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ (\mathsf{T}_\ell\ \mathsf{s}_1)^+) \times (\mathsf{s}^+ \to (\mathsf{T}_\ell\ \mathsf{s}_1)^+).\mathbf{m_2}$.

Have: $\mathbf{m_1'}[\boldsymbol{\rho_1}((\mathsf{T}_\ell\ \mathsf{s}_1)^+)/\boldsymbol\beta] \longmapsto^* \mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ (\mathsf{T}_\ell\ \mathsf{s}_1)^+) \times (\mathsf{s}^+ \to (\mathsf{T}_\ell\ \mathsf{s}_1)^+).\mathbf{m_2'}$.

Let $\boldsymbol\rho' = \boldsymbol\rho[\boldsymbol\beta \mapsto \boldsymbol\pi]$.

(4) Have: $(\mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ (\mathsf{T}_\ell \mathsf{s}_1)^+) \times (\mathsf{s}^+ \to (\mathsf{T}_\ell \mathsf{s}_1)^+).\mathbf{m_2}, \mathbf{\lambda x}{:}(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ (\mathsf{T}_\ell \mathsf{s}_1)^+) \times (\mathsf{s}^+ \to (\mathsf{T}_\ell \mathsf{s}_1)^+).\mathbf{m_2'}) \in \boldsymbol{\mathcal{E}}[\![((\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol\beta) \times (\mathsf{s}^+ \to \boldsymbol\beta)) \to \boldsymbol\beta]\!]_{\boldsymbol\rho'}^{\boldsymbol\Sigma}$.

Let $\mathbf{m} = \langle \mathbf{pf}[\![\ell \preceq \mathsf{T}_\ell\ \mathsf{s}_1]\!], \eta_{\mathbf{k}}^{\ell,\mathsf{s}} \rangle$.

Note that if $\ell \not\sqsubseteq \zeta$, $\mathbf{R} = Atom\,[(\mathsf{T}_\ell\ \mathsf{s}_1)^+]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$

(5) Therefore: $(\mathbf{m}, \mathbf{m}) \in \boldsymbol{\mathcal{E}}[\![(\boldsymbol{\alpha_\preceq}\ \ \boldsymbol{\alpha_\ell}\ \ \boldsymbol\beta) \times (\mathsf{s}^+ \to \boldsymbol\beta)]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$.

Have: $(\mathbf{m_2}[\mathbf{m}/\mathbf{x}], \mathbf{m_2'}[\mathbf{m}/\mathbf{x}]) \in \boldsymbol{\mathcal{E}}[\![\boldsymbol\beta]\!]_{\boldsymbol\rho'}^{\boldsymbol\Sigma}$ from 4 and 5.

(6) Have: $\forall \mathbf{m'}, \mathbf{m''}$ such that If $(\mathbf{m_2}[\mathbf{m}/\mathbf{x}], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m'}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ and $(\mathbf{m_2'}[\mathbf{m}/\mathbf{x}], \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\mathbf{m''}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$, then $(\mathbf{m'}, \mathbf{m''}) \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}_1^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ by definition of $\mathbf{R}$ and by assumption $\ell \sqsubseteq \zeta$.

Have: $\exists \hat{\mathbf{m}}_\mathbf{2}.(\mathbf{m_1}[(\mathsf{T}_\ell\ \mathsf{s}_1)^+/\boldsymbol\beta]\ \mathbf{m}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \hat{\mathbf{m}}_\mathbf{2}) \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol\rho}^{\boldsymbol\Sigma}$ and $(\mathsf{e}_2, \hat{\mathbf{m}}_\mathbf{2}) \in \mathcal{E}_\zeta^+[\![\mathsf{s}_1]\!]_{\boldsymbol{\rho_1}}$ by 2.

Have: $\exists \hat{\mathbf{m}}_2'.(\mathbf{m}_1'[(\mathsf{T}_\ell\ \mathsf{s}_1)^+/\boldsymbol{\beta}]\ \mathbf{m}, \eta_{\mathbf{k}}^{\ell,\mathsf{s}}\ \hat{\mathbf{m}}_2') \in \boldsymbol{\mathcal{E}}[\![(\mathsf{T}_\ell\ \mathsf{s}_1)^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$ and $(\mathsf{e}_2', \hat{\mathbf{m}}_2') \in \mathcal{E}_\zeta^+[\![\mathsf{s}_1]\!]_{\boldsymbol{\rho}_2}$ by 3.
Instantiating 6 with $\hat{\mathbf{m}}_2$ and $\hat{\mathbf{m}}_2'$: $(\hat{\mathbf{m}}_2, \hat{\mathbf{m}}_2') \in \boldsymbol{\mathcal{E}}[\![\mathsf{s}_1^+]\!]_{\boldsymbol{\rho}}^{\boldsymbol{\Sigma}}$
Proceed by induction (*1b*): $(\mathsf{e}_2, \mathsf{e}_2') \in \mathcal{E}[\![\mathsf{s}_1]\!]_\zeta$

□

Lemma 5.39 (Equivalence of Substitutions)
Let $\boldsymbol{\rho} = [\![\mathcal{L}_\ell^+]\!]_\zeta^{\boldsymbol{\Sigma}}$, $(\gamma_1, \boldsymbol{\gamma}_1) \in \mathcal{G}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\rho}_1}$, *and* $(\gamma_2, \boldsymbol{\gamma}_2) \in \mathcal{G}_\zeta^+[\![\mathsf{s}]\!]_{\boldsymbol{\rho}_2}$, *then:*

1. *If* $(\gamma_1, \gamma_2) \in \mathcal{G}[\![\mathsf{s}]\!]_\zeta$, *then* $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \in \boldsymbol{\mathcal{G}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}}$

2. *If* $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \in \boldsymbol{\mathcal{G}}[\![\mathsf{s}^+]\!]_{\boldsymbol{\rho}}^{\mathbf{D};\mathbf{G}}$, *then* $(\gamma_1, \gamma_2) \in \mathcal{G}[\![\mathsf{s}]\!]_\zeta$

Proof

Follows from Part 1 of Lemma 5.38. □

The following final theorem requires a few uninteresting extensions:

$\boxed{\vdash \gamma : \Gamma \rightsquigarrow \boldsymbol{\gamma}}$ where $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+ \vdash \boldsymbol{\gamma} : \Gamma^+$

$$\frac{}{\vdash \emptyset : \cdot \rightsquigarrow \emptyset} \qquad \frac{\cdot \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m} \qquad \vdash \gamma : \Gamma \rightsquigarrow \boldsymbol{\gamma}}{\vdash \gamma\,[\mathsf{x} \mapsto \mathsf{e}] : \Gamma, \mathsf{x} : \mathsf{s} \rightsquigarrow \boldsymbol{\gamma}\,[\mathbf{x} \mapsto \mathbf{m}]}$$

Figure 31: DCC to F_ω: Subst. Translation

Corollary 5.40 (Translation of Substitions)
Let $\boldsymbol{\delta} = \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \ \mid\ \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_\preceq \mapsto \hat{\boldsymbol{\alpha}}_\preceq\}$, $\boldsymbol{\gamma}_\sqsubseteq = [\![\mathcal{L}_\sqsubseteq^+]\!]$, *and* $\boldsymbol{\gamma}_\preceq = [\![\preceq^+]\!]$.
If $\vdash \gamma : \Gamma$ *then* $\vdash \gamma : \Gamma \rightsquigarrow \boldsymbol{\gamma}$ *and* $(\gamma, \boldsymbol{\gamma}_\sqsubseteq(\boldsymbol{\gamma}_\preceq(\boldsymbol{\gamma}))) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_{\boldsymbol{\delta}}$

$\boxed{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\gamma} : \Gamma^+ \Uparrow \gamma}$ where $\vdash \gamma : \Gamma$

$$\frac{}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \emptyset : \cdot \Uparrow \emptyset} \qquad \frac{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \mid \cdot \vdash \mathbf{m} : \mathsf{s}^{\hat{+}} \Uparrow \mathsf{e} \qquad \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\gamma} : \Gamma^+ \Uparrow \gamma}{\boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\gamma}\,[\mathbf{x} \mapsto \mathbf{m}] : \Gamma^+, \mathbf{x} : \mathsf{s}^{\hat{+}} \Uparrow \gamma\,[\mathsf{x} \mapsto \mathsf{e}]}$$

Figure 32: Back-translation: F_ω substitutions

Corollary 5.41 (Back-translation of Substitutions)
Let $\boldsymbol{\delta} = \{\boldsymbol{\alpha}_\ell \mapsto \hat{\boldsymbol{\alpha}}_\ell \ \mid\ \ell \in \mathcal{L}_\ell\} \cup \{\boldsymbol{\alpha}_\preceq \mapsto \hat{\boldsymbol{\alpha}}_\preceq\}$
If $\boldsymbol{\Sigma} \vdash \boldsymbol{\gamma} : \Gamma^+$ *then* $\exists \gamma.\ \boldsymbol{\Sigma}; \mathbf{G_k}; \Gamma^{\hat{+}} \vdash \boldsymbol{\gamma} : \Gamma^+ \Uparrow \gamma$ *and* $(\gamma, \boldsymbol{\gamma}) \in \mathcal{G}_\zeta^+[\![\Gamma]\!]_{\boldsymbol{\delta}}$

Theorem 5.42 **($\approx_\zeta$ Preservation and Reflection)**
Let $\Gamma \vdash \mathsf{e_1} : \mathsf{s} \rightsquigarrow \mathbf{m_1}$ *and* $\Gamma \vdash \mathsf{e_2} : \mathsf{s} \rightsquigarrow \mathbf{m_2}$*.*

1. *If* $\Gamma \vdash \mathsf{e_1} \approx_\zeta \mathsf{e_2} : \mathsf{s}$*, then* $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m_1} \approx_\zeta \mathbf{m_2} : \mathsf{s}^+$*.*
2. *If* $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m_1} \approx_\zeta \mathbf{m_2} : \mathsf{s}^+$*, then* $\Gamma \vdash \mathsf{e_1} \approx_\zeta \mathsf{e_2} : \mathsf{s}$*.*

Proof

Part 1 follows from Lemma 5.23 (Correctness of Translation), Lemma 5.41 (Back-translation of Substitutions) and part 1a of Lemma 5.38 (Preservation and Reflection of Closed Terms).

Part 2 follows from Lemma 5.23 (Correctness of Translation), Lemma 5.40 (Translation of Substitutions), and part 1b of Lemma 5.38 (Preservation and Reflection of Closed Terms). □

Theorem 5.43 **(Noninterference: Indirect Proof)**
If $\Gamma \vdash \mathsf{e} : \mathsf{s}$ *then* $\Gamma \vdash \mathsf{e} \approx_\zeta \mathsf{e} : \mathsf{s}$

Proof

By correctness of the translation:
$\Gamma \vdash \mathsf{e} : \mathsf{s} \rightsquigarrow \mathbf{m}$, $\Gamma \vdash \mathsf{e} \simeq \mathbf{m} : \mathsf{s}$, and $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} : \mathsf{s}^+$.

By parametricity, $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx \mathbf{m} : \mathsf{s}^+$

Since $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx \mathbf{m} : \mathsf{s}^+$ quantifies over *all* $\mathbf{D}$, $\mathbf{G}$, $\boldsymbol{\rho}$, $\boldsymbol{\gamma_1}$, $\boldsymbol{\gamma_2}$, the *particular* implementations we use in the observer-sensitive definition work, so: $\mathcal{L}_\ell^+; \mathcal{L}_\sqsubseteq^+, \preceq^+, \Gamma^+ \vdash \mathbf{m} \approx_\zeta \mathbf{m} : \mathsf{s}^+$

By reflection, $\Gamma \vdash \mathsf{e} \approx_\zeta \mathsf{e} : \mathsf{s}$. □

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