Summary of Proof Techniques

To prove a goal of the form:

1. $P \to Q$:
   (a) Assume P is true and prove Q.
   (b) Prove the contrapositive; that is, assume that Q is false and prove that P is false.
2. $\neg P$:
   (a) Reexpress as a positive statement.
   (b) Use proof by contradiction; that is, assume that P is true and try to reach a contradiction.
3. $P \wedge Q$:
   Prove P and Q separately. In other words, treat this as two separate goals: P, and Q.
4. $P \vee Q$:
   (a) Use proof by cases. In each case, either prove P or prove Q.
   (b) Assume P is false and prove Q, or assume Q is false and prove P.
5. $P \leftrightarrow Q$:
   Prove $P \to Q$ and $Q \to P$, using the methods listed under part 1.
6. $\forall x P(x)$:
   Let x stand for an arbitrary object, and prove $P(x)$. (If the letter x already stands for something in the proof, you will have to use a different letter for the arbitrary object.)
7. $\exists x P(x)$:
   Find a value of x that makes $P(x)$ true. Prove $P(x)$ for this value of x.
8. $\exists! x P(x)$:
   (a) Prove $\exists x P(x)$ (existence) and $\forall y \forall z((P(y) \wedge P(z)) \to y = z)$ (uniqueness).
   (b) Prove the equivalent statement $\exists x(P(x) \wedge \forall y(P(y) \to y = x))$.

9. $\forall n \in \mathbb{N} P(n)$:
   (a) Mathematical Induction: Prove $P(0)$ (base case) and $\forall n \in \mathbb{N}(P(n) \to P(n+1))$ (induction step).
   (b) Strong Induction: Prove $\forall n \in \mathbb{N}[(\forall k < n P(k)) \to P(n)]$.

To use a given of the form:

1. $P \to Q$:
   (a) If you are also given P, or you can prove that P is true, then you can conclude that Q is true.
   (b) Use the contrapositive: If you are given or can prove that Q is false, you can conclude that P is false.
2. ~~$\neg P$:~~
   ~~(a) Reexpress as a positive statement.~~
   ~~(b) In a proof by contradiction, you can reach a contradiction by proving P.~~
3. $P \wedge Q$:
   Treat this as two givens: P, and Q.
4. $P \vee Q$:
   (a) Use proof by cases. In case 1 assume P is true, and in case 2 assume Q is true.
   (b) If you are also given that P is false, or you can prove that P is false, you can conclude that Q is true. Similarly, if you know that Q is false you can conclude that P is true.
5. $P \leftrightarrow Q$:
   Treat this as two givens: $P \to Q$, and $Q \to P$.
6. $\forall x P(x)$:
   You may plug in any value, say a, for x, and conclude that $P(a)$ is true.
7. $\exists x P(x)$:
   Introduce a new variable, say x_0, into the proof, to stand for a particular object for which $P(x_0)$ is true.
8. $\exists! x P(x)$:
   Introduce a new variable, say x_0, into the proof, to stand for a particular object for which $P(x_0)$ is true. You may also assume that $\forall y \forall z((P(y) \wedge P(z)) \to y = z)$.

Techniques that can be used in any proof:

1. ~~Proof by contradiction: Assume the goal is false and derive a contradiction.~~
2. Proof by cases: Consider several cases that are *exhaustive*, that is, that include all the possibilities. Prove the goal in each case.