Proving that something is false

CPSC 509: Programming Language Principles

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So far in class we have mostly been proving that something is true, for example that "There is a program in Vapid1 with undefined result."

Sometimes, we want to prove that something is *not* true though, for example, "There is no Vapid 0 program with undefined result." Proving something of the form "not P" is common, so we should make sure we understand how to do that.

Suppose I have some proposition P. I may want to prove that "P is false" or "not P." In symbolic notation, this is written

$$\neg P.$$

To prove something of this form, the standard practice is to prove that "If P is true then *absurdity* follows." In logic, we represent absurdity[1] with the symbol $\bot$, which is typically given the name "bottom." So for our purposes, $\neg P$ is just an abbreviation for $P \Rightarrow \bot$. The intuition is that if P is true then something is really broken in the world.

Though we haven't explicitly stated it before, there are a lot of things that we already know are not true, meaning that they imply $\bot$. For instance, we know that the atom true is not the same as the atom false. In our typical mathematical notation we write this as

$$\text{true} \neq \text{false}$$

But this is shorthand for

$$\neg(\text{true} = \text{false}) \quad \text{(i.e., "it is not the case that true} = \text{false.")}$$

and *that* is shorthand for

$$(\text{true} = \text{false}) \Rightarrow \bot \quad \text{(i.e., "if true} = \text{false then the world is broken.")c}$$

We can use knowledge of this proposition to prove that something is false about our language of Boolean Expressions:

Proposition 1. *true* $\not\Downarrow$ *false*.

Rewriting this symbolically, we are proving that $\neg(\text{true} \Downarrow \text{false})$, i.e., that $(\text{true} \Downarrow \text{false}) \Rightarrow \bot$. We are proving an implication and we already know how to do that: assume the premise and use that to prove the conclusion.

Proof. Suppose that true $\Downarrow$ false. By inversion on $t \Downarrow v$, we that for all values v, if true $\Downarrow v$ then $v =$ true. Specializing this for our assumption, it follows that false $=$ true. But that's absurd (i.e., we apply (true $=$ false) $\Rightarrow \bot$ to deduce absurdity $\bot$).

Thus we've proven that it's absurd that true $\Downarrow$ false or rather true $\Downarrow$ false $\implies \bot$. ☐

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[1] You may have heard the word "contradiction" as a synonym for absurdity. For technical reasons I'm avoiding that word, and I also want to assure you that what I am about to demonstrate is *not* "proof by contradiction."